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DESIGN SENSITIVITY ANALYSIS OF COUPLED THERMOVISCOELASTIC SYSTEMS

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Abstract—Both the direct differentiation and adjoint variable methods for design sensitivity analysis of transient dynamic, arbitrarily nonlinear thermoviscoelastic coupled systems are presented in this paper. The approach is based on the thermodynamic description of simple materials due to Coleman. Large strains as well as arbitrary material nonlinearities are accounted for in the derivations. The domain parametrization or reference volume concept is used allowing a uniform treatment of both the shape and sizing design sensitivity analysis. Analytical examples demonstrating the use of the derived equations are given. This approach provides a solid basis for discretization of the developed equations for subsequent use in numerical computations.

NOTATION

$a_i, a_i^1, a_i^2, a_i^3, a_i^4$	arguments of functions f, f_1, f_2, f_3, f_4
A, A_0	surface in the deformed and undeformed configurations
A_1, A_2	specified constants
$\hat{A}$	linear functional of variations
$A^*_{kl}(\lambda_{ij})$	linear functional of Lagrange multipliers
b	design parameter
B	solid body under consideration
$\hat{B}$	linear functional of variations
$B^*(\lambda_{ij})$	linear functional of Lagrange multipliers
c	specific heat at constant strain
C	arbitrary part of the deformed configuration
$d_{ij} = \dfrac{\partial y_i}{\partial x_j}$	elements of the inverse Jacobian matrix
D	reference domain
$\dfrac{D}{Dt}$	material derivative
$D_{\varepsilon_{ij}}, D_T$	partial differential operators with respect to the current values of strains and temperature
$D_{ij}(t-\tau)$	kernel in thermoviscoelastic constitutive relation
e	internal energy per unit mass
$f^0, f_1^0, f_2^0, f_3^0, f_4^0$	functions in the performance functional in the undeformed configuration
f, f_1, f_2, f_3, f_4	functions in the performance functional in the reference configuration
F_i	components of body force
g_i	temperature gradient
G	unknown function in analytical examples
$G_1, G_2, \tilde{G}_1, \tilde{G}_2$	specific constants
$G_{ijkl}(t-\tau, t-s)$	kernels in thermoviscoelastic constitutive relation
$G_{ij}^A(s), G^B(s)$	integrating functions in Stieltjes integral
h	convective coefficient
H	unknown function in analytical examples
$H_1, H_2, \tilde{H}_1, \tilde{H}_2$	specified constants
$J, \tilde{J}$	Jacobians of the mappings from the reference domain to the undeformed configuration and from the undeformed to the deformed configuration
$J_\Gamma, \tilde{J}_\Gamma$	Jacobians of the mappings of the boundary in the reference domain onto the boundary in the undeformed configuration and the undeformed configuration onto the boundary in the deformed configuration
K	specified constant
$K(t-\tau)$	kernel in thermoviscoelastic constitutive relation
K_s, K_c	sine and cosine Fourier transforms of thermoviscoelastic kernel

$K^A_{kl}(t)$, $K^B(t)$	kernels in constitutive relations
L	wavelength parameter
$m(t-\tau, t-s)$	kernel in thermoviscoelastic constitutive relation
n_k, $\tilde{n}_i$, n^0_j	components of the external normal to the boundary in the reference domain, in the deformed configuration, and in the undeformed configuration
$n(t-\tau)$	kernel in thermoviscoelastic constitutive relation
P	performance functional
P_a	augmented functional
$\tilde{q}_i$, q_i	components of heat flux in deformed and undeformed configurations
$\bar{\tilde{q}}$, $\bar{q}$	prescribed heat flux at the boundary in the deformed configuration and undeformed configurations
Q	internal heat generation per unit volume
Q_0	intensity of internal heat source
$\underset{\tau=0}{\overset{\infty}{Q_i}}$	constitutive functional for heat flux in the undeformed configuration
$\underset{\tau=0}{\overset{\infty}{Q'_i}}$	constitutive functional for heat flux in the reference domain
r	heat supply per unit mass
R	radiation coefficient
s	time
$\tilde{S}_i$, S_i	components of surface traction in the deformed and undeformed configurations
$\bar{\tilde{S}}_i$, $\bar{S}_i$	components of the prescribed surface traction in the deformed and undeformed configurations
t	time
t_0	terminal time
T	absolute temperature
T_0	initial temperature
u_i	components of displacement
$\bar{u}_i$	components of the prescribed displacement at the boundary
$u_{0i}(x_k)$	initial distribution of displacements
$\dot{u}_{0i}(x_k)$	initial distribution of velocities
$\tilde{u}_1$	sensitivity of u_1
V^0, $\tilde{V}$	deformed and undeformed configurations
$w_{rp} = u_{r;p}$	displacement gradients in reference domain
x_1, x_2, x_3	Cartesian coordinates in the undeformed configuration
y_1, y_2, y_3	Cartesian coordinates in the reference domain
z_1, z_2, z_3	Cartesian coordinates in the deformed configuration
α	coefficient of thermal expansion
$\alpha(t-\tau)$	kernel in thermoviscoelastic constitutive relation
α_s, α_c	sine and cosine Fourier transforms of kernel
$\beta(t-\tau)$	kernel in thermoviscoelastic constitutive relation
β_1	solution of adjoint differential equation in example problem
β_2	solution of adjoint differential equation in example problem
β_s, β_c	sine and cosine Fourier transforms of kernel
γ	specified constant
Γ, Γ^0, $\tilde{\Gamma}$	boundaries of the reference domain, undeformed and deformed configurations
Γ_u, Γ_s, Γ_T, Γ_q, Γ_h, Γ_R	parts of boundary of the reference domain with prescribed displacements, tractions, temperature, heat flux, convective boundary conditions, and radiative boundary conditions
$\Gamma^0_u, \Gamma^0_s, \Gamma^0_T, \Gamma^0_q, \Gamma^0_h, \Gamma^0_R$	parts of boundary of the undeformed configuration with prescribed displacements, tractions, temperature, heat flux, convective boundary conditions, and radiative boundary conditions
δ	variation of a quantity
$\hat{\delta}(t-\tau)$, $\hat{\delta}(x_1-y)$	delta functions
$\delta_{\varepsilon_{ij}}$, δ_T, δ_{g_k}	Frechet differentials with respect to the past histories of strains, temperature, and temperature gradients
$\delta_{\varepsilon_{ij}} \underset{\tau=0}{\overset{\infty}{Q'^*_n}}$	adjoint functional derived from the heat flux constitutive functional
$\delta_T \underset{\tau=0}{\overset{\infty}{Q'^*_n}}$	adjoint functional derived from the heat flux constitutive functional
$\delta_{g_m} \underset{\tau=0}{\overset{\infty}{Q'^*_i}}$	adjoint functional derived from the heat flux constitutive functional
$\delta_{\varepsilon_{ij}} \underset{\tau=0}{\overset{\infty}{\Xi^*_{st}}}$	adjoint functional derived from the stress constitutive functional
$\delta_T \underset{\tau=0}{\overset{\infty}{\Xi^*_{ij}}}$	adjoint functional derived from the stress constitutive functional
$\delta_{\varepsilon_{ij}} \underset{\tau=0}{\overset{\infty}{\Theta^*}}$	adjoint functional derived from the entropy constitutive functional
$\delta_T \underset{\tau=0}{\overset{\infty}{\Theta^*}}$	adjoint functional derived from the entropy constitutive functional
$\delta_T \underset{\tau=0}{\overset{\infty}{\Psi^*}}$	adjoint functional derived from the free energy constitutive functional

$\delta_{\varepsilon_{ij}} \overset{\infty}{\underset{\tau=0}{\Psi^*}}$	adjoint functional derived from the free energy constitutive functional
ε	parameter in Frechet differential calculations
ε_{ij}	components of the Green-Lagrange strain tensor
ε_{ijk}	alternating tensor
ε^r_{st}	past strain history
ζ	specified constant
η	specific entropy
$\overset{\infty}{\underset{\tau=0}{\Theta}}$	constitutive functional for entropy
κ	specified constant
λ	Lamé coefficient
$\lambda(t-\tau)$	kernel in thermoviscoelastic constitutive relation
$\lambda^i_1, \lambda_2, \lambda^{ij}_3, \lambda^i_4, \lambda^{ij}_5, \lambda_6, \lambda_7, \lambda^i_8, \lambda^i_9, \lambda_{10}, \lambda_{11}, \lambda_{12}, \lambda_{13}$	Lagrange multipliers
μ	Lamé coefficient
$\rho, \tilde{\rho}$	mass densities in the undeformed and deformed configurations
σ_{ij}	components of the second Piola–Kirchhoff stress tensor
σ^r_{ij}	past stress history
τ	time
$\phi(t-\tau)$	kernel in thermoviscoelastic constitutive relation
$\Phi_{ij}(t-\tau, t-s)$	kernel in thermoviscoelastic constitutive relation
$\xi, \xi_1, \xi_2, \xi_3, \xi_4$	specified constants
$\overset{\infty}{\underset{\tau=0}{\Xi_{ij}}}$	constitutive functional for stresses
ψ	free energy function
$\overset{\infty}{\underset{\tau=0}{\Psi}}$	constitutive functional for free energy
ω	internal dissipation
Ω	constitutive functional for internal dissipation
,	derivative with respect to coordinates in the undeformed configuration
;	derivative with respect to coordinates in the reference domain.

1. INTRODUCTION

A number of publications have appeared recently on Design Sensitivity Analysis (DSA) of thermoelastic structures (Meric, 1986a, b; Dems, 1987; Dems and Mroz, 1987; Meric, 1987, 1988, 1990; Tortorelli *et al.*, 1989; Hou *et al.*, 1990, 1991). These publications deal with thermoelastic systems, that is, they discount hysteretic effects which are significant in many materials like rubbers and polymers. Two of the above articles (Meric, 1988; Tortorelli *et al.*, 1991) deal with fully coupled linear thermoelastic structures, whereas all others omit the term in the energy equation responsible for the full coupling of the mechanical and thermal processes. This term may not be very important for a certain class of thermoelastic structures (Boley and Weiner, 1960), but may become significant for structures exhibiting hysteresis. Tortorelli *et al.* (1989, 1991) considered adjoint sensitivities for thermoelastic structures with nonlinearly elastic constitutive relations and a nonlinear analogue of Fourier law. All the papers mentioned deal with structures undergoing infinitesimal strains.

Due to the continually increasing CPU and memory capacity of modern computers, arbitrarily nonlinear thermoviscoelastic coupled calculations based on the finite element method (FEM) are becoming more feasible for engineering structures. The next logical step in this direction is to use FEM for structural optimization which requires DSA. In addition, DSA may be valuable by itself for evaluation of the structural response due to uncertainties in design or manufacturing. DSA can be implemented either as a finite difference scheme, or as a special module built into the FEM software making use of the intermediate results in FEM. The latter approach is much more efficient in terms of CPU time which is critical for successful implementation of structural optimization. That is why there is a need for development of DSA approaches for arbitrarily nonlinear fully coupled systems with internal dissipation.

The present paper deals with arbitrarily nonlinear thermoviscoelastic systems. The derivations are based on a consistent thermodynamical approach due to Coleman (1964a, b). The only two constitutive functionals needed in this approach are: the free energy functional and the generalized Fourier law, i.e. a functional relating the heat flux to the temperature gradients, the temperature and strains. Both direct differentiation sensitivity equations and adjoint sensitivity equations are obtained. It is shown that both groups of equations are of linear hereditary type similar to linear viscoelasticity. Analytical examples illustrating the use of the developed equations are given.

2. FIELD EQUATIONS

We consider a continuous solid body B which may be subjected to external forces, applied displacements, temperatures and external heat fluxes. At time $t = -0$, external forces, displacements and fluxes are not applied, and the absolute temperature distribution is uniform having a value of T_0. The body B, at this time moment, occupies the undeformed configuration in the space V^0 with the boundary Γ^0. As the external excitations such as forces, displacements, temperatures, etc. begin to act on the body B, it moves and deforms in space and at time t occupies the deformed configuration $\tilde{V}$ with the boundary $\tilde{\Gamma}$.

We introduce a Cartesian coordinate system fixed in space and measure coordinates with respect to it. Denote the coordinates of a point in the undeformed configuration as x_1, x_2, x_3 and the coordinates of a point in the deformed configuration as z_1, z_2, z_3.

The motion, deformation and the thermal state of the body B should satisfy five physical laws:

(a) Balance of Momentum

$$\frac{D}{Dt}\int_C \tilde{\rho}\dot{u}_i \,\mathrm{d}C = \int_C \tilde{\rho}F_i \,\mathrm{d}C + \int_A \tilde{S}_i \,\mathrm{d}A; \tag{1}$$

(b) Balance of Angular Momentum

$$\frac{D}{Dt}\int_C \tilde{\rho}\varepsilon_{ijk}z_j\dot{u}_k \,\mathrm{d}C = \int_C \tilde{\rho}\varepsilon_{ijk}z_jF_k \,\mathrm{d}C + \int_A \varepsilon_{ijk}z_j\tilde{S}_k \,\mathrm{d}A; \tag{2}$$

(c) Conservation of Mass

$$\frac{D}{Dt}\int_C \tilde{\rho}\,\mathrm{d}C = 0; \tag{3}$$

(d) First Law of Thermodynamics: Conservation of Energy

$$\frac{D}{Dt}\left(\frac{1}{2}\int_C \tilde{\rho}\dot{u}_i\dot{u}_i \,\mathrm{d}C + \int_C \tilde{\rho}e\,\mathrm{d}C\right) = \int_C \tilde{\rho}r\,\mathrm{d}C - \int_A \tilde{q}_i\tilde{n}_i \,\mathrm{d}A + \int_C \tilde{\rho}F_i\dot{u}_i \,\mathrm{d}C + \int_A \tilde{S}_i\dot{u}_i \,\mathrm{d}A; \tag{4}$$

(e) Second Law of Thermodynamics: Clausius–Duhem inequality

$$\frac{D}{Dt}\int_C \tilde{\rho}\eta\,\mathrm{d}C - \int_C \tilde{\rho}\frac{r}{T}\mathrm{d}C + \int_A \frac{\tilde{q}_i\tilde{n}_i}{T}\mathrm{d}A \geqslant 0. \tag{5}$$

Integration in the above equations is over C—an arbitrary part of the deformed configuration $\tilde{V}$, A is the C boundary, and D/Dt denotes the material derivative. Summation over repeated indices is assumed everywhere. The variables in (1)–(5) are defined as: $\tilde{\rho}$—mass density, u_i—components of displacement, F_i—components of body force, $\tilde{S}_i$—components of surface traction, ε_{ijk}—alternating tensor, e—internal energy per unit mass, r—heat supply per unit mass, $\tilde{q}_i$—components of heat flux, $\tilde{n}_i$—components of the exterior unit normal to the surface, η—specific entropy, T—absolute temperature.

In the following derivations we will adopt the Lagrangian point of view that is commonly used in the description of solid bodies. Equations (1)–(5) can be easily reformulated in terms of the undeformed configuration for Lagrangian description, and then, with appropriate smoothness assumptions the corresponding local equations are obtained:

$$\rho \ddot{u}_i = (z_{i,j}\sigma_{kj})_{,k} + \rho F_i, \tag{6}$$

$$\sigma_{ij} = \sigma_{ji}, \tag{7}$$

$$\rho = \tilde{\rho}\tilde{J}, \tag{8}$$

$$\rho \dot{e} = \rho r - q_{i,i} + \sigma_{lm}\dot{\varepsilon}_{lm}, \tag{9}$$

$$\rho T \dot{\eta} \geq -q_{i,i} + \frac{q_i T_{,i}}{T} + \rho r. \tag{10}$$

Equation (7) is obtained from (2) with the additional assumption of absence of concentrated couples.

Equation (9) can also be written as:

$$\rho T \dot{\eta} - \rho r - \omega + q_{i,i} = 0, \tag{11}$$

where ω is the internal dissipation defined as:

$$\omega = \sigma_{ij}\dot{\varepsilon}_{ij} - \rho(\dot{\psi} + \eta \dot{T}), \tag{12}$$

where ψ is the free energy defined as:

$$\psi = e - \eta T. \tag{13}$$

Commas in all the above equations denote partial differentiation with respect to the coordinates in the undeformed configuration. The quantities used in eqns (6)–(10) are: ρ—mass density in the undeformed configuration, σ_{ij}—components of the second Piola–Kirchhoff stress tensor (Fung, 1965), $\tilde{J} = |\partial z_i/\partial x_j|$—Jacobian of the transformation from the undeformed to the deformed configuration, q_i—components of the heat flux in the undeformed configuration, ε_{ij}—components of the Green–Lagrange strain tensor expressed in terms of the displacement components as:

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{l,i}u_{l,j}). \tag{14}$$

Components of the heat flux in the undeformed configuration can be expressed in terms of the heat flux components in the deformed configuration. The components of the normal exterior to a surface in the deformed configuration $\tilde{n}_i$ are related to the components of the normal in the undeformed configuration n_j^0 through the following formula (Fung, 1965):

$$\tilde{n}_i \, \mathrm{d}A = \tilde{J} \frac{\partial x_j}{\partial z_i} n_j^0 \, \mathrm{d}A_0, \tag{15}$$

where A_0 is the surface in the undeformed configuration corresponding to the surface A in the deformed configuration. Therefore, it is clear from eqn (4) that the components of the heat flux in the undeformed configuration will be:

$$q_j = \tilde{J}\tilde{q}_i \frac{\partial x_j}{\partial z_i}. \tag{16}$$

Equations (6)–(10), (14) have to be solved with appropriate boundary conditions, initial conditions and constitutive equations.

3. THERMODYNAMICS AND CONSTITUTIVE EQUATIONS

We consider simple viscoelastic materials. Such materials are characterized by relationships in which response at a material point depends on the total history of the dependent quantities, but only at the same material point. Dependent quantities such as free energy function, stresses, heat flux and entropy in constitutive relations for such material may depend on strains, temperatures and temperature gradients, but be independent of strain gradients. Relations between the thermodynamic potentials for such materials were developed by Coleman (1964a,b) and are also summarized in Oden (1972) and Truesdell and Noll (1965). All quantities such as stresses, temperatures, strains, etc. are functions of time in addition to the spatial coordinates x_i:

$$\sigma_{ij} = \sigma_{ij}(x_s, t-\tau); \quad T = T(x_i, t-\tau) \tag{17}$$

with $0 \leqslant \tau < \infty$. Following Coleman, they are called the total histories. Coleman (1964b) introduced the so-called past histories σ^r_{ij}, T^r, etc. which are total histories excluding the values at the current time t, or, in other words, all the quantities are restricted to the open interval $0 < \tau < \infty$. The constitutive equations consist of four functionals: free energy ψ, stresses σ_{ij}, the heat flux components q_i, and the specific entropy η which are written in terms of the past histories ε^r_{ij}, T^r, the current values $\varepsilon_{ij} = \varepsilon_{ij}(x_l, t)$, $T = T(x_l, t)$ and the temperature gradients $g_i = T_{,i}$ as:

$$\psi = \mathop{\Psi}_{\tau=0}^{\infty} (\varepsilon^r_{ij}, T^r, \varepsilon_{lm}, T), \tag{18}$$

$$\sigma_{ij} = \mathop{\Xi_{ij}}_{\tau=0}^{\infty} (\varepsilon^r_{st}, T^r, \varepsilon_{lm}, T), \tag{19}$$

$$q_i = \mathop{Q_i}_{\tau=0}^{\infty} (u_{j,m}, \varepsilon^r_{st}, T^r, g^r_k, T, g_l), \tag{20}$$

$$\eta = \mathop{\Theta}_{\tau=0}^{\infty} (\varepsilon^r_{ij}, T^r, \varepsilon_{lm}, T). \tag{21}$$

In (18)–(21), the lower limit 0 and the upper limit ∞ in the functionals stress the fact that the functionals depend on the entire history of the variables. The functionals are defined on a Hilbert space with a norm introduced in Coleman (1964a) which is based on the fading memory assumption.

We note that the functional for the heat flux in (20) depends on the displacement gradients in view of (16). The heat flux, however, depends only on the current values of the displacement gradients and not on their entire history.

It was shown by Coleman (1964a) that as a consequence of the Second Law of Thermodynamics formally expressed by inequality (10), the functional for the free energy Ψ in (16) is independent of the temperature gradients g_i, and that the functionals for the stresses Ξ, the specific entropy Θ, and the internal dissipation Ω are determined through the free energy functional as:

$$\mathop{\Xi_{ij}}_{\tau=0}^{\infty} (\varepsilon^r_{st}, T^r, \varepsilon_{lm}, T) = D_{\varepsilon_{ij}} \mathop{\Psi}_{\tau=0}^{\infty} (\varepsilon^r_{st}, T^r, \varepsilon_{lm}, T), \tag{22}$$

$$\mathop{\Theta}_{\tau=0}^{\infty} (\varepsilon^r_{ij}, T^r, \varepsilon_{lm}, T) = -D_T \mathop{\Psi}_{\tau=0}^{\infty} (\varepsilon^r_{ij}, T^r, \varepsilon_{lm}, T), \tag{23}$$

$$\Omega = -\rho[\delta_{\varepsilon_{st}} \mathop{\Psi}_{\tau=0}^{\infty} (\varepsilon^r_{ij}, T^r, \varepsilon_{lm}, T|\dot{\varepsilon}^r_{st}) + \delta_T \mathop{\Psi}_{\tau=0}^{\infty} (\varepsilon^r_{ij}, T^r, \varepsilon_{lm}, T|\dot{T}^r)], \tag{24}$$

where operators $D_{\varepsilon_{ij}}$ and D_T are partial differential operators with respect to the current values of ε_{ij} and T; $\delta_{\varepsilon_{ij}}$ and δ_T are Frechet differentials with respect to the past histories of strains and temperature. $\dot{\varepsilon}^r_{st}$ and $\dot{T}^r$ are calculated as:

$$\dot{\varepsilon}^r_{st} = -\frac{\mathrm{d}\varepsilon_{st}(t-\tau)}{\mathrm{d}\tau}, \tag{25}$$

$$\dot{T}^r = -\frac{\mathrm{d}T(t-\tau)}{\mathrm{d}\tau}. \tag{26}$$

The vertical line in (24) indicates that the functionals are linear with respect to the following arguments or in other words specifying the variation direction of the arguments.

The Frechet differentials in (24) can be calculated as:

$$\delta_{\varepsilon_{st}} \overset{\infty}{\underset{\tau=0}{\Psi}} (\varepsilon^r_{ij}, T^r, \varepsilon_{lm}, T|\dot{\varepsilon}^r_{st}) = \lim_{\varepsilon\to 0} \frac{\partial \overset{\infty}{\underset{\tau=0}{\Psi}} (\varepsilon^r_{st}+\varepsilon\dot{\varepsilon}^r_{st}, T^r, \varepsilon_{lm}, T)}{\partial\varepsilon}, \tag{27}$$

$$\delta_T \overset{\infty}{\underset{\tau=0}{\Psi}} (\varepsilon^r_{ij}, T^r, \varepsilon_{lm}, T|\dot{T}^r) = \lim_{\varepsilon\to 0} \frac{\partial \overset{\infty}{\underset{\tau=0}{\Psi}} (\varepsilon^r_{st}, T^r+\varepsilon\dot{T}^r, \varepsilon_{lm}, T)}{\partial\varepsilon}. \tag{28}$$

Therefore, for the description of simple viscoelastic materials in thermomechanical problems, only two constitutive functionals such as free energy and the functional for heat flux are required. The functionals for stresses, entropy and internal dissipation are then determined from (22)–(24).

Everywhere in the following we will use constitutive functionals as functionals of total histories without splitting them into the past histories and the current values to avoid extra variables in the notation. That is we write:

$$\psi = \overset{\infty}{\underset{\tau=0}{\Psi}} (\varepsilon_{st}, T, b), \tag{29}$$

$$\sigma_{ij} = \overset{\infty}{\underset{\tau=0}{\Xi_{ij}}} (\varepsilon_{st}, T, b), \tag{30}$$

$$\eta = \overset{\infty}{\underset{\tau=0}{\Theta}} (\varepsilon_{st}, T, b), \tag{31}$$

$$q_i = \overset{\infty}{\underset{\tau=0}{Q_i}} (u_{j,m}, \varepsilon_{st}, T, g_k, b). \tag{32}$$

We included design parameter b in the above equations emphasizing the fact that the constitutive equations may depend on it.

4. BOUNDARY AND INITIAL CONDITIONS

The boundary Γ^0 in the undeformed configuration is subdivided into two sets of disjoint domains:

$$\Gamma^0 = \Gamma^0_u \cup \Gamma^0_s = \Gamma^0_T \cup \Gamma^0_q \cup \Gamma^0_h \cup \Gamma^0_R, \tag{33}$$

where
Γ^0_u is the boundary with specified displacements,
Γ^0_s is the boundary with specified surface tractions,
Γ^0_T is the boundary with specified temperature,
Γ^0_q is the boundary with specified heat flux,
Γ^0_h is the boundary with specified convective boundary conditions,
Γ^0_R is the boundary with specified radiation boundary conditions.

The boundary conditions are written as:

(1) On Γ_u^0

$$u_i = \bar{u}_i, \tag{34}$$

where $\bar{u}_i$ is the ith component of the prescribed displacement.

(2) On Γ_s^0

$$z_{i,j}\sigma_{kj}n_k^0 = \bar{S}_i, \tag{35}$$

where n_k^0 is the kth component of the outward normal, and $\bar{S}_i$ is the ith component of the prescribed surface traction. $\bar{S}_i$ is defined on the boundary surface in the undeformed configuration. Its magnitude is referred to the original area of the boundary, but it acts in the same direction as the surface traction in the deformed configuration. Since surface tractions $\bar{\bar{S}}_i$ are normally prescribed in the deformed configuration, the $\bar{S}_i$ may be displacement dependent because

$$\bar{S}_i = \bar{\bar{S}}_i \tilde{J}_\Gamma, \tag{36}$$

where $\tilde{J}_\Gamma$ is the Jacobian of the mapping of the boundary in the undeformed configuration onto the boundary in the deformed configuration and given by the following expression (Haug *et al.*, 1986):

$$\tilde{J}_\Gamma = J' \sqrt{\frac{\partial x_i}{\partial z_j} n_i^0 \frac{\partial x_k}{\partial z_j} n_k^0}. \tag{37}$$

Note that follower forces are accounted for in this formulation since $\bar{S}_i$ follows the directions in the deformed configuration.

(3) On Γ_T^0

$$T = \bar{T}, \tag{38}$$

where $\bar{T}$ is the prescribed temperature.

(4) On Γ_q^0

$$-q_i n_i^0 = \bar{q}, \tag{39}$$

where $\bar{q}$ is the prescribed heat flux. Again, as in the case of the prescribed surface tractions, the prescribed heat flux $\bar{q}$ is considered to be displacement dependent. Particularly, this includes the case when the heat flux $\bar{\bar{q}}$ is prescribed in the deformed configuration, which is generally the case. Then,

$$\bar{q} = \bar{\bar{q}}\tilde{J}_\Gamma. \tag{40}$$

(5) On Γ_h^0

$$-q_i n_i^0 = h(T_f - T), \tag{41}$$

where h is the convective coefficient, and considered to be displacement dependent consistently with what was said about the surface tractions and the prescribed heat flux. T_f is the fluid temperature.

(6) On Γ_R^0

$$-q_i n_i^0 = R(T_R^4 - T^4), \tag{42}$$

where R is the radiation coefficient which depends on the Stefan-Boltzman constant, the emissivity, absorptivity and the geometric view factors. T_R is the temperature of the radiative source.

The initial conditions at time $t = 0$ are:

$$u_i(x_k, 0) = u_{0i}(x_k), \tag{43}$$

$$\dot{u}_i(x_k, 0) = \dot{u}_{0i}(x_k), \tag{44}$$

$$T(x_k, 0) = T_0, \tag{45}$$

where $u_{0i}(x_k)$, $\dot{u}_{0i}(x_k)$ and T_0 are the initial distributions of displacements, velocities and temperature, respectively.

5. GENERALIZED PERFORMANCE FUNCTIONAL AND REFERENCE VOLUME CONCEPT

The aim of design sensitivity analysis is typically to find design variations or partial derivatives of displacements, stresses, temperatures, energies or functionals of the above quantities over a certain period of time, or at the specific time instant t. Such quantities can be written in a general way as a performance functional P:

$$P = \int_0^{t_0} \left[\int_{V^0} f^0(u_i, u_{r,p}, \dot{u}_j, \ddot{u}_k, \varepsilon_{lm}, \sigma_{st}, \eta, T, q_\alpha, b)\, \mathrm{d}V^0 + \int_{\Gamma_u^0} f_1^0(S_i, b)\, \mathrm{d}\Gamma + \int_{\Gamma_s^0} f_2^0(u_i, b)\, \mathrm{d}\Gamma + \int_{\Gamma_T^0} f_3^0(q_\alpha, b)\, \mathrm{d}\Gamma + \int_{\Gamma_q^0 \cup \Gamma_h^0 \cup \Gamma_R^0} f_4^0(T, b)\, \mathrm{d}\Gamma \right] \mathrm{d}t, \tag{46}$$

where b is the design parameter, S_i is the surface traction expressed through the second Piola–Kirchhoff stresses according to (35).

Depending on the form of the functions f^0, f_1^0, f_2^0, f_3^0 and f_4^0 in (46), various quantities can be derived. Particularly, use of the Dirac delta functions in space and/or in time leads to functionals such as displacements, stresses, temperatures, etc. at a given point in space and/or in time. Also, note that presence of the displacement gradients in (46) allows calculations of the quantities in the deformed configuration like the heat flux or the Cauchy stress. Some simple performance functionals are shown later in the example section.

The problem under consideration is how to calculate the variation of the functional P given by (46) due to variation of the design parameter b.

To calculate the variation of the functional P we will use the reference volume or the domain parametrization concept (Haber, 1987; Cardoso and Arora, 1988), to unify sizing and shape design sensitivity analysis. A fixed reference domain D is introduced. It stays unchanged during both deformation and design variation processes. The same fixed Cartesian coordinate system is used to describe the reference domain as well as the deformed and the undeformed configurations. Every point in the reference domain D is characterized by coordinates y_1, y_2, y_3 and there is a one-to-one correspondence between the reference domain and the undeformed configurations. The mapping of the reference domain onto the undeformed configuration is written:

$$x_i = x_i(y_1, y_2, y_3, b). \tag{47}$$

Dependence on the design parameter b in (47) emphasizes the fact the undeformed configuration is design-dependent. The reference domain, however, is not. The reference domain is defined in such a way that the Jacobian of the mapping $J = |\partial x_i/\partial y_j|$ is not zero. Therefore, the inverse mapping exists:

$$y_i = y_i(x_1, x_2, x_3, b). \tag{48}$$

In the above described mapping, the boundary Γ^0 and its parts $\Gamma_u^0, \Gamma_s^0, \Gamma_T^0, \Gamma_q^0, \Gamma_h^0, \Gamma_R^0$ in the undeformed configuration correspond to the boundary Γ and its parts Γ_u, Γ_s, Γ_T, Γ_q, Γ_h, Γ_R in the reference domain.

Since we will use elements of the inverse Jacobian matrix extensively, we denote them as:

$$d_{ij} = \frac{\partial y_i}{\partial x_j}. \tag{49}$$

The generalized performance functional P in the reference domain is:

$$P = \int_0^{t_0} \left[\int_D Jf(u_i, w_{rp}, \dot{u}_j, \ddot{u}_k, \varepsilon_{lm}, \sigma_{st}, \eta, T, q_\alpha, b)\,\mathrm{d}D + \int_{\Gamma_u} J_\Gamma f_1(S_i, b)\,\mathrm{d}\Gamma + \int_{\Gamma_s} J_\Gamma f_2(u_i, b)\,\mathrm{d}\Gamma + \int_{\Gamma_T} J_\Gamma f_3(q_\alpha, b)\,\mathrm{d}\Gamma + \int_{\Gamma_q \cup \Gamma_h \cup \Gamma_r} J_\Gamma f_4(T, b)\,\mathrm{d}\Gamma \right] \mathrm{d}t, \tag{50}$$

where $w_{rp} = u_{r;p}$ are the displacement gradients in the reference domain, and semicolon from now on denotes partial derivative with respect to the reference domain coordinates. The Green–Lagrange strain components in the reference domain are:

$$\varepsilon_{lm} = \tfrac{1}{2}(u_{l;k}d_{km} + u_{m;k}d_{kl} + u_{n;k}d_{kl}u_{n;j}d_{jm}). \tag{51}$$

J_Γ is the Jacobian of the mapping of the boundary in the reference domain onto the boundary in the undeformed configuration calculated similarly to (37):

$$J_\Gamma = J\sqrt{d_{ij}n_i d_{kj}n_k} \tag{52}$$

with n_k being the components of the external normal to the boundary in the reference domain. $S_i = z_{i;l}d_{lj}\sigma_{kj}Jd_{mk}n_m$ are the components of the surface traction in the reference domain. Functions f, f_1, f_2, f_3, f_4 in (50) can be easily expressed through functions $f^0, f_1^0, f_2^0, f_3^0, f_4^0$ in (46) knowing the relations between the displacement gradients in the undeformed configuration and in the reference domain.

By changing the configuration under consideration, the conservation laws (1)–(5) can be easily transformed into the reference domain to give the equations of motion, the energy equation, and the Clausius–Duhem inequality in the reference domain:

$$J\rho\ddot{u}_i = (z_{i;s}d_{sj}\sigma_{kj}Jd_{lk})_{;l} + J\rho F_i, \tag{53}$$

$$J\rho\dot{e} = J\rho r - (q_i J d_{ji})_{;j} + J\sigma_{lm}\dot{\varepsilon}_{lm}, \tag{54}$$

$$J\rho T\dot{\eta} \geqslant -(q_i J d_{ji})_{;j} + \frac{q_i J d_{ji} T_{;j}}{T} + J\rho r. \tag{55}$$

Equation (7) does not change—the stress tensor is still symmetric. Equation (8) does not provide new information unless the mass density is given in the reference domain.

The constitutive equations for free energy, stresses and entropy (29)–(31) do not change their form, but the constitutive equation (32) for the heat flux does. It happens because the displacement gradients in the reference domain are expressed in terms of the displacement gradients in the undeformed configuration as follows:

$$u_{i,j} = u_{i;l}d_{lj}. \tag{56}$$

Therefore, the constitutive equation (32) in the reference domain can be written as:

$$q_i = \underset{\tau=0}{\overset{\infty}{Q'_i}}\,(u_{j;m}, \varepsilon_{st}, T, g_k, b), \tag{57}$$

where the functional $\underset{\tau=0}{\overset{\infty}{Q'_i}}$ is readily expressed in terms of the functional $\underset{\tau=0}{\overset{\infty}{Q_i}}$.

The boundary conditions on Γ_s, Γ_q, Γ_h will change to:

$$z_{i;s}d_{sj}\sigma_{kj}Jd_{lk}n_l = \bar{S}_i J_\Gamma, \tag{58}$$

$$-q_i J d_{ji} n_j = \bar{q} J_\Gamma, \tag{59}$$

$$-q_i J d_{ji} n_j = h(T_f - T) J_\Gamma, \tag{60}$$

$$-q_i J d_{ji} n_j = R(T_R^4 - T^4) J_\Gamma. \tag{61}$$

The boundary conditions on Γ_u, Γ_T stay unchanged.

The reference domain, however, is not exactly a configuration used for referring all quantities to it as in the case of the deformed or undeformed configurations. It is rather a convenient means to parametrize the structure for subsequent design changes. Mathematically it means that the independent variables are changed from x_1, x_2, x_3 to y_1, y_2, y_3, but it does not mean that stresses, heat flux or surface traction are calculated per unit area of the reference domain. They are still calculated per unit area of the undeformed or deformed configurations, but are simply functions of the new variables.

We note that the reference volume concept is naturally translated into isoparametric finite elements where the reference domain is one which is a union of reference domains of individual elements. Thus, the reference volume concept can be readily used in numerical analysis of thermomechanical problems.

6. SENSITIVITY ANALYSIS BY DIRECT DIFFERENTIATION METHOD (DDM)

Using this approach, we take variation of the functional in (50):

$$\begin{aligned}\delta P = \int_0^{t_0}\Bigg[&\int_D\left(\delta J f + J\left(\frac{\partial f}{\partial b}\delta b + \frac{\partial f}{\partial a_i}\delta a_i\right)\right)\mathrm{d}D + \int_{\Gamma_u}\left(\delta J_\Gamma f_1 + J_\Gamma\left(\frac{\partial f_1}{\partial b}\delta b + \frac{\partial f_1}{\partial a_i^1}\delta a_i^1\right)\right)\mathrm{d}\Gamma \\ &+ \int_{\Gamma_s}\left(\delta J_\Gamma f_2 + J_\Gamma\left(\frac{\partial f_2}{\partial b}\delta b + \frac{\partial f_2}{\partial a_i^2}\delta a_i^2\right)\right)\mathrm{d}\Gamma + \int_{\Gamma_T}\left(\delta J_\Gamma f_3 + J_\Gamma\left(\frac{\partial f_3}{\partial b}\delta b + \frac{\partial f_3}{\partial a_i^3}\delta a_i^3\right)\right)\mathrm{d}\Gamma \\ &+ \int_{\Gamma_q\cup\Gamma_h\cup\Gamma_r}\left(\delta J_\Gamma f_4 + J_\Gamma\left(\frac{\partial f_4}{\partial b}\delta b + \frac{\partial f_4}{\partial a_i^4}\delta a_i^4\right)\right)\mathrm{d}\Gamma\Bigg]\mathrm{d}t, \end{aligned} \tag{62}$$

where $a_i, a_i^1, a_i^2, a_i^3, a_i^4$ denote the arguments of functions f, f_1, f_2, f_3, f_4 other than design parameter b, that is displacements, strains, temperatures, etc. Variations of arguments $a_i, a_i^1, a_i^2, a_i^3, a_i^4$ or field variables are not known and are obtained by solving the varied field equations along with the varied boundary conditions, strain compatibility and constitutive equations in the reference domain.

The equations for variations to be solved are:

$$\begin{aligned}J\rho\delta\ddot{u}_i + \delta(J\rho)\ddot{u}_i = (\delta z_{i;s}d_{sj}\sigma_{kj}Jd_{lk})_{;l} + (z_{i;s}d_{sj}\delta\sigma_{kj}Jd_{lk})_{;l} + [z_{i;s}\sigma_{kj}\delta(d_{sj}Jd_{lk})]_{;l} \\ + \delta(J\rho)F_i + J\rho\delta F_i, \end{aligned} \tag{63}$$

$$\delta(J\rho)\dot{e}+J\rho\delta\dot{e} = \delta(J\rho)r+J\rho\delta r-(\delta q_i Jd_{ji})_{;j}-[q_i\delta(Jd_{ji})]_{;j}+\delta J\sigma_{lm}\dot{\varepsilon}_{lm}+J\delta\sigma_{lm}\dot{\varepsilon}_{lm}+J\sigma_{lm}\delta\dot{\varepsilon}_{lm}, \quad (64)$$

$$\delta\varepsilon_{lm} = \tfrac{1}{2}(\delta u_{l;k}d_{km}+u_{l;k}\delta d_{km}+\delta u_{m;k}d_{kl}+u_{m;k}\delta d_{kl}+\delta u_{n;k}d_{kl}u_{n;j}d_{jm} + u_{n;k}\delta d_{kl}u_{n;j}d_{jm}+u_{n;k}d_{kl}\delta u_{n;j}d_{jm}+u_{n;k}d_{kl}u_{n;j}\delta d_{jm}), \quad (65)$$

$$\delta e = \delta_{\varepsilon_{st}}\overset{\infty}{\underset{\tau=0}{\Psi}}(\varepsilon_{lm},T,b\,|\,\delta\varepsilon_{st})+\delta_T\overset{\infty}{\underset{\tau=0}{\Psi}}(\varepsilon_{lm},T,b\,|\,\delta T)+D_b\overset{\infty}{\underset{\tau=0}{\Psi}}(\varepsilon_{lm},T,b)\delta b+\delta\eta T+\eta\delta T, \quad (66)$$

$$\delta\sigma_{ij} = \delta_{\varepsilon_{st}}\overset{\infty}{\underset{\tau=0}{\Xi_{ij}}}(\varepsilon_{lm},T,b\,|\,\delta\varepsilon_{st})+\delta_T\overset{\infty}{\underset{\tau=0}{\Xi_{ij}}}(\varepsilon_{lm},T,b\,|\,\delta T)+D_b\overset{\infty}{\underset{\tau=0}{\Xi_{ij}}}(\varepsilon_{lm},T,b)\delta b, \quad (67)$$

$$\delta\eta = \delta_{\varepsilon_{st}}\overset{\infty}{\underset{\tau=0}{\Theta}}(\varepsilon_{lm},T,b\,|\,\delta\varepsilon_{st})+\delta_T\overset{\infty}{\underset{\tau=0}{\Theta}}(\varepsilon_{lm},T,b\,|\,\delta T)+D_b\overset{\infty}{\underset{\tau=0}{\Theta}}(\varepsilon_{lm},T,b)\delta b, \quad (68)$$

$$\delta q_i = D_{w_{rp}}\overset{\infty}{\underset{\tau=0}{Q'_i}}(w_{rp},\varepsilon_{lm},T,g_k,b)\delta w_{rp}+\delta_T\overset{\infty}{\underset{\tau=0}{Q'_i}}(w_{rp},\varepsilon_{lm},T,g_k,b\,|\,\delta T)+ \delta_{\varepsilon_{st}}\overset{\infty}{\underset{\tau=0}{Q'_i}}(w_{rp},\varepsilon_{lm},T,g_m,b\,|\,\delta\varepsilon_{st})+\delta_{g_k}\overset{\infty}{\underset{\tau=0}{Q'_i}}(w_{rp},\varepsilon_{lm},T,g_m,b\,|\,\delta g_k) + D_b\overset{\infty}{\underset{\tau=0}{Q'_i}}(w_{rp},\varepsilon_{lm},T,g_k,b)\delta b. \quad (69)$$

If one chooses to use the energy equation in the form (11) with the internal dissipation ω given by (12), eqns (64) and (66) are replaced with the following:

$$\delta(J\rho)T\dot{\eta}+J\rho\delta T\dot{\eta}+J\rho T\delta\dot{\eta} = \delta(J\rho)r+J\rho\delta r-(\delta q_i Jd_{ji})_{;j}-[q_i\delta(Jd_{ji})]_{;j}+\delta(J\omega), \quad (70)$$

$$\delta(J\omega) = \delta J\sigma_{lm}\dot{\varepsilon}_{lm}+J\delta\sigma_{lm}\dot{\varepsilon}_{lm}+J\sigma_{lm}\delta\dot{\varepsilon}_{lm}-\delta(J\rho)(\dot{\psi}+\eta\dot{T})-J\rho(\delta\dot{\psi}+\delta\eta\dot{T}+\eta\delta\dot{T}). \quad (71)$$

To calculate Frechet differentials $\delta_{\varepsilon_{ij}}$, δ_T in (66)–(69) one has to know the "directions" of the variations. They are defined by defining the primary sensitivities, i.e.

$$\delta u_i = \frac{\partial u_i}{\partial b}\delta b, \quad (72)$$

$$\delta T = \frac{\partial T}{\partial b}\delta b. \quad (73)$$

Then, the Frechet differentials in (66)–(69) are calculated as:

$$\delta_{\varepsilon_{st}}\overset{\infty}{\underset{s=0}{f}}(\varepsilon_{lm},T,b\,|\,\delta\varepsilon_{st}) = \lim_{\varepsilon\to 0}\frac{\partial\overset{\infty}{\underset{s=0}{f}}(\varepsilon_{lm}+\varepsilon\delta\varepsilon_{lm},T,b)}{\partial\varepsilon}, \quad (74)$$

$$\delta_T\overset{\infty}{\underset{s=0}{f}}(\varepsilon_{lm},T,b\,|\,\delta T) = \lim_{\varepsilon\to 0}\frac{\partial\overset{\infty}{\underset{s=0}{f}}(\varepsilon_{lm},T+\varepsilon\delta T,b)}{\partial\varepsilon}, \quad (75)$$

$$\delta_{g_k}\overset{\infty}{\underset{s=0}{f}}(w_{rp},T,g_k,b\,|\,\delta g_k) = \lim_{\varepsilon\to 0}\frac{\partial\overset{\infty}{\underset{s=0}{f}}(w_{rp},T,g_k+\varepsilon\delta g_k,b)}{\partial\varepsilon}, \quad (76)$$

where $\delta\varepsilon_{st}$ is defined by (65) with δu_i given by (72), and δT given by (73).

The Frechet differentials

$$\delta_{\varepsilon_{st}} \overset{\infty}{\underset{\tau=0}{\Psi}} (\varepsilon_{lm}, T, b \,|\, \delta\varepsilon_{st}), \quad \delta_T \overset{\infty}{\underset{\tau=0}{\Psi}} (\varepsilon_{lm}, T, b \,|\, \delta T), \quad \delta_{\varepsilon_{st}} \overset{\infty}{\underset{\tau=0}{\Xi_{ij}}} (\varepsilon_{lm}, T, b \,|\, \delta\varepsilon_{st}),$$

$$\delta_T \overset{\infty}{\underset{\tau=0}{\Xi_{ij}}} (\varepsilon_{lm}, T, b \,|\, \delta T), \quad \delta_{\varepsilon_{st}} \overset{\infty}{\underset{\tau=0}{\Theta}} (\varepsilon_{lm}, T, b \,|\, \delta\varepsilon_{st}), \quad \delta_T \overset{\infty}{\underset{\tau=0}{\Theta}} (\varepsilon_{lm}, T, b \,|\, \delta T),$$

$$\delta_T \overset{\infty}{\underset{\tau=0}{Q_i'}} (w_{rp}, \varepsilon_{st}, T, g_k, b \,|\, \delta T), \quad \delta_{\varepsilon_{st}} \overset{\infty}{\underset{\tau=0}{Q_i'}} (w_{rp}, \varepsilon_{st}, T, g_m, b \,|\, \delta\varepsilon_{st}), \quad \delta_{g_k} \overset{\infty}{\underset{\tau=0}{Q_i'}} (w_{rp}, \varepsilon_{st}, T, g_m, b \,|\, \delta g_k)$$

are linear functionals of the variations $\delta\varepsilon_{st}$, δT which are assumed to be continuous functions of time. Therefore, according to the Riesz representation theorem (Riesz and Nagy, 1955), they can be written as a Stieltjes integral:

$$\hat{A}(\delta\varepsilon_{st}) = \int_0^\infty \delta\varepsilon_{kl}(t-s)\,\mathrm{d}G^A_{kl}(t,s),$$

$$\hat{B}(\delta T) = \int_0^\infty \delta T(t-s)\,\mathrm{d}G^B(t,s), \tag{77}$$

where $\hat{A}$, $\hat{B}$ are linear functionals which can be one of the above Frechet differentials. Integration in (77) is along variable s. The integrating functions $G^A_{ij}(t,s)$, $G^B(t,s)$ may depend on stresses, temperatures, design parameters, and calculated in the process of the Frechet differential calculations.

Similarly to Christensen (1982), assuming that G^A_{kl} and G^B have first continuous derivatives, taking strain and temperature histories $\varepsilon_{ij}(t) = 0$, $T(t) = 0$ for $t < 0$, and changing variable $\tau = t-s$, (77) can be written as:

$$\hat{A}(\delta\varepsilon_{st}) = G^A_{kl}(t,0)\delta\varepsilon_{kl}(t) + \int_0^t K^A_{kl}(t,\tau)\delta\varepsilon_{kl}(\tau)\,\mathrm{d}\tau,$$

$$\hat{B}(\delta T) = G^B(t,0)\delta T(t) + \int_0^t K^B(t,\tau)\delta T(\tau)\,\mathrm{d}\tau, \tag{78}$$

where

$$K^A_{kl}(t,\tau) = -\frac{\mathrm{d}G^A_{kl}(t,t-\tau)}{\mathrm{d}\tau}, \quad K^B(t,\tau) = -\frac{\mathrm{d}G^B(t,t-\tau)}{\mathrm{d}\tau}.$$

We see, therefore, that eqns (66)–(69) are linear equations of hereditary type, but contrary to linear viscoelastic ones their kernels depend on two variables t and τ and not just on their difference. It means that the equations for sensitivity analysis do not obey the requirement of time translation invariance as follows from the axiom of material frame indifference (Truesdell and Noll, 1965), for the constitutive functionals of the real physical systems.

Varying boundary conditions (34), (38), (58)–(61), boundary conditions for the design variations are obtained as:

(1) On Γ_u

$$\delta u_i = \delta\bar{u}_i. \tag{79}$$

(2) On Γ_s

$$\delta z_{i;s}\, d_{sj}\sigma_{kj} J d_{lk} n_l + z_{i;s}\sigma_{kj}\delta(d_{sj} J d_{lk}) n_l + z_{i;s} d_{sj}\delta\sigma_{kj} J d_{lk} n_l = \delta\bar{S}_i J_\Gamma + \bar{S}_i \delta J_\Gamma. \tag{80}$$

(3) On Γ_T

$$\delta T = \delta\bar{T}. \tag{81}$$

(4) On Γ_q

$$-\delta q_i J d_{ji} n_j = \delta \bar{q} J_\Gamma + \bar{q}\delta J_\Gamma + q_i \delta(J d_{ji}) n_j. \tag{82}$$

(5) On Γ_h

$$-\delta q_i J d_{ji} n_j = \delta h(T_f - T) J_\Gamma + h(\delta T_f - \delta T) J_\Gamma + h(T_f - T)\delta J_\Gamma + q_i \delta(J d_{ji}) n_j. \tag{83}$$

(6) On Γ_R

$$-\delta q_i J d_{ji} n_j = \delta R(T_R^4 - T^4) J_\Gamma + 4R(T_R^3 \delta T_R - T^3 \delta T) J_\Gamma + R(T_R^4 - T^4)\delta J_\Gamma + q_i \delta(J d_{ji}) n_j. \tag{84}$$

We note that variations $\delta \bar{S}_i$, $\delta \bar{q}$ and δh are sums of variations with respect to the displacements and the design parameter, and the displacement parts of these variations add to the left-hand side, whereas the design parts add to the right-hand side.

Similarly, varying the initial conditions, we obtain at $t = 0$:

$$\delta u_i = \delta u_{0i}, \tag{85}$$

$$\delta \dot{u}_i = \delta \dot{u}_{0i}, \tag{86}$$

$$\delta T = \delta T_0. \tag{87}$$

Note that since the reference domain is used, the normal is fixed during the design variation, and its variation is not present in the equations.

Thus, a boundary value problem for design variations of the field variables is defined. If the original boundary value problem has a unique solution and is continuously differentiable with respect to the design parameter, the solution of the design sensitivity problem also exists and is unique. This can be observed from the fact that the derivative of the original solution with respect to a parameter will be the solution of the design sensitivity equations in variations. Note that the design sensitivity equations (63)–(69) with the boundary conditions (79)–(84) and the initial conditions (85)–(87) are linear even though the original equations may be nonlinear. But in spite of the linearity of the design sensitivity equations and boundary conditions, they depend on displacements, strains, temperatures, etc. of the original problem appearing as variable coefficients in the equations and boundary conditions complicating the solution process. Variation of the functional (62) is found by substituting the solution of (63)–(69), (79)–(84), (85)–(87) into (62). If variations of a large number of functionals have to be computed with respect to a small number of design variables, then DDM has certain advantages because for all the functionals with the same design variables the same DDM equations are solved, and then the solutions are simply substituted into different expressions for different functionals. On the other hand, if there are few functionals and many design variables DDM may not be very efficient because a new set of the design sensitivity equations has to be solved for every design variable. For the latter case the adjoint variable method described in the next section will be more efficient.

7. SENSITIVITY ANALYSIS BY ADJOINT VARIABLE METHOD (AVM)

The adjoint variable method provides a means to avoid explicit calculations of the design variations of the field variables. Instead, by introducing the adjoint variables which are Lagrange multipliers (LM), design variation of a given functional is calculated directly by solving equations for the LM and substituting them into the formula for variation of the performance functional. The equations for the LM are obtained from the condition that all terms in the augmented functional having implicit variations vanish (Arora and Cardoso, 1992). Therefore, variation of the performance functional will be expressed in terms of the LM and the explicit variations.

First, we contract every equation describing the problem with its own LM, integrate it, and add it to the performance fuctional to get the augmented functional P_a:

$$
\begin{aligned}
P_a = \int_0^{t_0} \Bigg[& \int_D Jf(u_i, w_{rp}, \dot{u}_j, \ddot{u}_k, \varepsilon_{lm}, \sigma_{st}, \eta, T, q_\alpha, b)\,\mathrm{d}D + \int_{\Gamma_u} J_\Gamma f_1(S_i, b)\,\mathrm{d}\Gamma \\
& + \int_{\Gamma_s} J_\Gamma f_2(u_i, b)\,\mathrm{d}\Gamma + \int_{\Gamma_T} J_\Gamma f_3(q_\alpha, b)\,\mathrm{d}\Gamma + \int_{\Gamma_q \cup \Gamma_h \cup \Gamma_r} J_\Gamma f_4(T, b)\,\mathrm{d}\Gamma \\
& + \int_D \Big[\lambda_1^i (J\rho\ddot{u}_i - (z_{i;s} d_{sj} \sigma_{kj} J d_{lk})_{;l} - J\rho F_i) + \lambda_2 (J\rho\dot{e} - J\rho r + (q_i J d_{ji})_{;j} - J\sigma_{lm}\dot{\varepsilon}_{lm}) \\
& + \lambda_3^{lm} (\varepsilon_{lm} - \tfrac{1}{2}(u_{l;k} d_{km} + u_{m;k} d_{kl} + u_{n;k} d_{kl} u_{n;j} d_{jm})) \\
& + \lambda_4^i \Big(q_i - \mathop{Q_i'}_{\tau=0}^{\infty} (w_{rp}, \varepsilon_{st}, T, g_k, b) \Big) + \lambda_5^{ij} \Big(\sigma_{ij} - \mathop{\Xi_{ij}}_{\tau=0}^{\infty} (\varepsilon_{st}, T, b) \Big) + \lambda_6 \Big(\eta - \mathop{\Theta}_{\tau=0}^{\infty} (\varepsilon_{st}, T, b) \Big) \\
& + \lambda_7 \Big(e - \mathop{\Psi}_{\tau=0}^{\infty} (\varepsilon_{st}, T, b) - \eta T \Big) \Big] \mathrm{d}V + \int_{\Gamma_u} \lambda_8^i (\delta\bar{u}_i - \delta u_i)\,\mathrm{d}\Gamma \\
& + \int_{\Gamma_s} \lambda_9^i (\bar{S}_i J_\Gamma - z_{i;s} d_{sj} \sigma_{kj} J d_{lk} n_l)\,\mathrm{d}\Gamma + \int_{\Gamma_T} \lambda_{10} (\bar{T} - T)\,\mathrm{d}\Gamma \\
& + \int_{\Gamma_q} \lambda_{11} (\bar{q} J_\Gamma + q_i J d_{ji} n_j)\,\mathrm{d}\Gamma + \int_{\Gamma_h} \lambda_{12} (h(T_f - T) J_\Gamma + q_i J d_{ji} n_j)\,\mathrm{d}\Gamma \\
& + \int_{\Gamma_R} \lambda_{13} (R(T_R^4 - T^4) J_\Gamma + q_i J d_{ji} n_j)\,\mathrm{d}\Gamma \Bigg] \mathrm{d}t. \qquad (88)
\end{aligned}
$$

To obtain boundary conditions for the adjoint variables later on, we set

$$\lambda_9^i = -\lambda_1^i, \quad \lambda_{11} = \lambda_{12} = \lambda_{13} = -\lambda_2.$$

Since we consider design variations on the solutions of the field equations, the variations of all field equations times LM are zero, and $\delta P = \delta P_a$. Therefore, we can take variation of the augmented functional P_a to obtain the sought variation of P. Taking the variation of (88), part of the variation of the functional will split into integrals of LM variations times the field equations plus LM times the variation of the field equations. Knowing that the field equations are satisfied, we drop the first part of the variation obtaining the following expression for the variation of the augmented functional P_a:

$$
\begin{aligned}
\delta P_a = \int_0^{t_0} \Bigg[& \int_D \Big(\delta J f + J \Big(\frac{\partial f}{\partial b}\delta b + \frac{\partial f}{\partial a_i}\delta a_i \Big) \Big) \mathrm{d}D + \int_{\Gamma_u} \Big(\delta J_\Gamma f_1 + J_\Gamma \Big(\frac{\partial f_1}{\partial b}\delta b + \frac{\partial f_1}{\partial a_i^1}\delta a_i^1 \Big) \Big) \mathrm{d}\Gamma \\
& + \int_{\Gamma_s} \Big(\delta J_\Gamma f_2 + J_\Gamma \Big(\frac{\partial f_2}{\partial b}\delta b + \frac{\partial f_2}{\partial a_i^2}\delta a_i^2 \Big) \Big) \mathrm{d}\Gamma + \int_{\Gamma_T} \Big(\delta J_\Gamma f_3 + J_\Gamma \Big(\frac{\partial f_3}{\partial b}\delta b + \frac{\partial f_3}{\partial a_i^3}\delta a_i^3 \Big) \Big) \mathrm{d}\Gamma \\
& + \int_{\Gamma_q \cup \Gamma_h \cup \Gamma_r} \Big(\delta J_\Gamma f_4 + J_\Gamma \Big(\frac{\partial f_4}{\partial b}\delta b + \frac{\partial f_4}{\partial a_i^4}\delta a_i^4 \Big) \Big) \mathrm{d}\Gamma \\
& + \int_D [\lambda_1^i (J\rho\delta\ddot{u}_i + \delta(J\rho)\ddot{u}_i - (\delta z_{i;s} d_{sj} \sigma_{kj} J d_{lk})_{;l} - (z_{i;s} d_{sj} \delta\sigma_{kj} J d_{lk})_{;l} \\
& - [z_{i;s} \sigma_{kj} \delta(d_{sj} J d_{lk})]_{;l} - \delta(J\rho) F_i + J\rho\delta F_i) \\
& + \lambda_2 (\delta(J\rho)\dot{e} + J\rho\delta\dot{e} - \delta(J\rho) r - J\rho\delta r + (\delta q_i J d_{ji})_{;j} + [q_i \delta(J d_{ji})]_{;j}
\end{aligned}
$$

$$-\delta J\sigma_{lm}\dot{\varepsilon}_{lm}-J\delta\sigma_{lm}\dot{\varepsilon}_{lm}-J\sigma_{lm}\delta\dot{\varepsilon}_{lm})$$

$$+\lambda_3^{lm}(\delta\varepsilon_{lm}-\tfrac{1}{2}(\delta u_{l;k}d_{km}+u_{l;k}\delta d_{km}+\delta u_{m;k}d_{kl}+u_{m;k}\delta d_{kl}+\delta u_{n;k}d_{kl}u_{n;j}d_{jm}$$

$$+u_{n;k}\delta d_{kl}u_{n;j}d_{jm}+u_{n;k}d_{kl}\delta u_{n;j}d_{jm}+u_{n;k}d_{kl}u_{n;j}\delta d_{jm}))$$

$$+\lambda_4^i\left(\delta q_i-D_{w_{rp}}\mathop{Q_i'}_{\tau=0}^{\infty}(w_{rp},T,g_k,b)\delta w_{rp}-\delta_T\mathop{Q_i'}_{\tau=0}^{\infty}(w_{rp},T,g_k,b\,|\,\delta T)\right.$$

$$\left.-\delta_{g_k}\mathop{Q_i'}_{\tau=0}^{\infty}(w_{rp},T,g_m,b\,|\,\delta g_k)-D_b\mathop{Q_i'}_{\tau=0}^{\infty}(w_{rp},T,g_k,b)\delta b\right)$$

$$+\lambda_5^{ij}\left(\delta\sigma_{ij}-\delta_{\varepsilon_{st}}\mathop{\Xi_{ij}}_{\tau=0}^{\infty}(\varepsilon_{lm},T,b\,|\,\delta\varepsilon_{st})-\delta_T\mathop{\Xi_{ij}}_{\tau=0}^{\infty}(\varepsilon_{lm},T,b\,|\,\delta T)-D_b\mathop{\Xi_{ij}}_{\tau=0}^{\infty}(\varepsilon_{lm},T,b)\,\delta b\right)$$

$$+\lambda_6\left(\delta\eta-\delta_{\varepsilon_{st}}\mathop{\Theta}_{\tau=0}^{\infty}(\varepsilon_{lm},T,b\,|\,\delta\varepsilon_{st})-\delta_T\mathop{\Theta}_{\tau=0}^{\infty}(\varepsilon_{lm},T,b\,|\,\delta T)-D_b\mathop{\Theta}_{\tau=0}^{\infty}(\varepsilon_{lm},T,b)\delta b\right)$$

$$+\lambda_7\left(\delta e-\delta_{\varepsilon_{st}}\mathop{\Psi}_{\tau=0}^{\infty}(\varepsilon_{lm},T,b\,|\,\delta\varepsilon_{st})-\delta_T\mathop{\Psi}_{\tau=0}^{\infty}(\varepsilon_{lm},T,b\,|\,\delta T)-D_b\mathop{\Psi}_{\tau=0}^{\infty}(\varepsilon_{lm},T,b)\delta b\right.$$

$$\left.\left.-T\delta\eta-\eta\delta T\right)\right]\mathrm{d}V+\int_{\Gamma_u}\lambda_8^i(\delta\bar{u}_i-\delta u_i)\,\mathrm{d}\Gamma$$

$$-\int_{\Gamma_s}\lambda_1^i\,(\delta(\bar{S}_iJ_\Gamma)-\delta z_{i;s}d_{sj}\sigma_{kj}Jd_{kl}n_l-z_{i;s}\sigma_{kj}\delta(d_{sj}Jd_{kl})n_l$$

$$-z_{i;s}d_{sj}\delta\sigma_{kj}Jd_{kl}n_l)\,\mathrm{d}\Gamma+\int_{\Gamma_T}\lambda_{10}(\delta T_0-\delta T)\,\mathrm{d}\Gamma-\int_{\Gamma_q}\lambda_2(\delta(\bar{q}J_\Gamma)$$

$$+\delta q_iJd_{ji}n_j+q_i\delta(Jd_{ji})n_j)\,\mathrm{d}\Gamma-\int_{\Gamma_h}\lambda_2\Big(\delta h(T_f-T)J_\Gamma+h(\delta T_f-\delta T)J_\Gamma$$

$$+h(T_f-T)\delta J_\Gamma+\delta q_iJd_{ji}n_j+q_i\delta(Jd_{ji}n_j)\,\mathrm{d}\Gamma$$

$$-\int_{\Gamma_R}\lambda_2(\delta R(T_R^4-T^4)J_\Gamma+4R(T_R^3\delta T_R-T^3\delta T)J_\Gamma$$

$$+R(T_R^4-T^4)\delta J_\Gamma+\delta q_iJd_{ji}n_j+q_i\delta(Jd_{ji})n_j)\,\mathrm{d}\Gamma\Big)\,\mathrm{d}t. \tag{89}$$

Now, the idea behind the AVM is to represent (89) as integrals of expressions depending on the LM λ_1^i through λ_{13} times the implicit variations δu_i, $\delta\varepsilon_{ij}$, $\delta\sigma_{ij}$, δe, δq_i, δT, $\delta\eta$. Then, by equating multipliers of the implicit variations to zero, the equations and the boundary conditions for LM λ_1^i through λ_{13} are obtained, thus eliminating all the implicit variations. The remaining terms will represent variation δP of the performance functional P, but will contain only the explicit variations.

To isolate the implicit variations, we perform integration by parts every time we find derivatives of the implicit variations both in time and in space. In the case of second derivatives, we perform integration by parts twice. We have also decided to drop the variations of the boundary conditions on Γ_u and Γ_T.

To isolate the implicit variations for the hereditary Frechet differentials in (89), we use the form in (78). We contract it with LM λ_{ij} and integrate it from 0 to t_0 as in (89):

$$\int_0^{t_0} \lambda_{ij}\hat{A}_{ij}(\delta\varepsilon_{st}) = \int_0^{t_0} \lambda_{ij}\left[G^A_{ijkl}(t,0)\delta\varepsilon_{kl}(t) + \int_0^t K^A_{ijkl}(t,\tau)\,\delta\varepsilon_{kl}(\tau)\,\mathrm{d}\tau\right]\mathrm{d}t,$$

$$\int_0^{t_0} \lambda_{ij}\hat{B}_{ij}(\delta T) = \int_0^{t_0} \lambda_{ij}\left[G^B_{ij}(t,0)\delta T(t) + \int_0^t K^B_{ij}(t,\tau)\delta T(\tau)\,\mathrm{d}\tau\right]\mathrm{d}t. \tag{90}$$

Changing the order of integration in the double integrals, we obtain:

$$\int_0^{t_0} \lambda_{ij}\hat{A}_{ij}(\delta\varepsilon_{st}) = \int_0^{t_0} [A^*_{kl}(\lambda_{ij})]\delta\varepsilon_{kl}(t)\,\mathrm{d}t,$$

$$\int_0^{t_0} \lambda_{ij}\hat{B}_{ij}(\delta T) = \int_0^{t_0} [B^*(\lambda_{ij})]\delta T(t)\,\mathrm{d}t, \tag{91}$$

where $A^*_{kl}(\lambda_{ij})$, $B^*(\lambda_{ij})$ are linear hereditary functionals of the following type:

$$A^*_{kl}(\lambda_{ij}) = G^A_{ijkl}(t,0)\lambda_{ij}(t) + \int_t^{t_0} K^A_{ijkl}(\tau,t)\lambda_{ij}(\tau)\,\mathrm{d}\tau,$$

$$B^*(\lambda_{ij}) = G^B_{ij}(t,0)\lambda_{ij}(t) + \int_t^{t_0} K^B_{ij}(\tau,t)\lambda_{ij}(\tau)\,\mathrm{d}\tau. \tag{92}$$

The hereditary functionals act now on the LM λ instead of the implicit variations $\delta\varepsilon_{st}$, or $\mathrm{d}T$, thus allowing them to be isolated.

After performing the mentioned integration by parts both in space and time, changing the order of integration for hereditary terms, and grouping terms, (89) takes the form:

$$\delta P_{\mathrm{a}} = \delta P_e + \int_0^{t_0}\left[\int_D\left[\left(J\rho\ddot{\lambda}^i_1 - (\lambda^i_{1;l}d_{sj}\sigma_{kj}Jd_{lk})_{;s} + (\lambda^{im}_3 d_{qm} + \lambda^{lm}_3 u_{i;p}d_{pl}d_{qm})_{;q}\right.\right.\right.$$
$$\left. + \left(\lambda^n_4 D_{w_{ip}} \overset{\infty}{\underset{\tau=0}{Q'_n}}\right)_{;p} - (Jf_{;w_{ip}})_{;p} + J(f_{;u_i} - \dot{f}_{;\dot{u}_i} + \ddot{f}_{;\ddot{u}_i})\right)\delta u_i$$
$$+ \left(J\frac{\mathrm{d}(\lambda_2\sigma_{ij})}{\mathrm{d}t} + \lambda^{ij}_3 - \delta_{\varepsilon_{ij}} \overset{\infty}{\underset{\tau=0}{Q'^*_n}}(\lambda^n_4) - \delta_{\varepsilon_{ij}} \overset{\infty}{\underset{\tau=0}{\Xi^*_{st}}}(\lambda^{st}_5) - \delta_{\varepsilon_{ij}} \overset{\infty}{\underset{\tau=0}{\Theta}}(\lambda_6) - \delta_{\varepsilon_{ij}} \overset{\infty}{\underset{\tau=0}{\Psi^*}}(\lambda_7) + Jf_{;\varepsilon_{ij}}\right)\delta\varepsilon_{ij}$$
$$+ (\tfrac{1}{2}[\lambda^j_{1;l}d_{lk} + \lambda^k_{1;l}d_{lj} + \lambda^i_{1;l}u_{i;s}(d_{sj}d_{lk} + d_{sk}d_{lj})] - J\lambda_2\dot{\varepsilon}_{kj} + \lambda^{kj}_5 + Jf_{,\sigma_{kj}})\delta\sigma_{kj}$$
$$+ (-J\rho\dot{\lambda}_2 + \lambda_7 + Jf_{;e})\delta e$$
$$+ \left(-\delta_T \overset{\infty}{\underset{\tau=0}{Q'^*_i}}(\lambda^i_4) + \left(\delta_{g_m} \overset{\infty}{\underset{\tau=0}{Q'^*_i}}(\lambda^i_4)\,d_{pm}\right)_{;p} - \delta_T \overset{\infty}{\underset{\tau=0}{\Xi^*_{ij}}}(\lambda^{ij}_5) - \delta_T \overset{\infty}{\underset{\tau=0}{\Theta^*}}(\lambda_6) - \delta_T \overset{\infty}{\underset{\tau=0}{\Psi^*}}(\lambda_7) - \eta\lambda_7 + Jf_{;T}\right)\delta T$$
$$\left. + (-\lambda_{2;j}Jd_{ji} + \lambda^i_4 + Jf_{;q_i})\delta q_i + (\lambda_6 - T\lambda_7 + Jf_{;\eta})\delta\eta\right]\mathrm{d}D + \int_{\Gamma_u}[-\lambda^i_1 + J_\Gamma f_{1;S_i}]\delta S_i\,\mathrm{d}\Gamma$$
$$+ \int_{\Gamma_s}\left[\lambda^i_{1;l}d_{sj}\sigma_{kj}Jd_{lk}n_s - (\lambda^{im}_3 d_{qm} + u_{i;p}\lambda^{lm}_3 d_{qm}d_{pl})n_q - \lambda^n_4\left(\overset{\infty}{\underset{\tau=0}{Q'_n}}\right)_{;w_{ip}} n_p + J_\Gamma f_{2;u_i} + Jf_{;w_{ip}}n_p\right]\delta u_i\,\mathrm{d}\Gamma$$
$$+ \int_{\Gamma_h}\left[-\delta_{g_m} \overset{\infty}{\underset{\tau=0}{Q'^*_i}}(\lambda^i_4)d_{pm}n_p - \lambda_2 J_\Gamma h + J_\Gamma f_{4;T}\right]\delta T\,\mathrm{d}\Gamma$$
$$+ \int_{\Gamma_R}\left[-\delta_{g_m} \overset{\infty}{\underset{\tau=0}{Q'^*_i}}(\lambda^i_4)d_{pm}n_p - 4\lambda_2 J_\Gamma RT^3 + J_\Gamma f_{4;T}\right]\delta T\,\mathrm{d}\Gamma$$

$$+\int_{\Gamma_q}\left[-\delta_{g_m}\overset{\infty}{\underset{\tau=0}{Q_i'^*}}(\lambda_4^i)d_{pm}n_p+J_\Gamma f_{4;T}\right]\delta T\,\mathrm{d}\Gamma+\int_{\Gamma_T}[\lambda_2 Jd_{ji}n_j+J_\Gamma f_{3;q_i}]\delta q_i\,\mathrm{d}\Gamma\Bigg]\mathrm{d}t$$

$$+\int_D[J(f_{;\dot{u}_i}-\dot{f}_{;\ddot{u}_i}-\rho\dot{\lambda}_1^i)_{t=t_0}\delta u_i+J(\rho\lambda_1^i+f_{;\ddot{u}_i})_{t=t_0}\delta\dot{u}_i+\lambda_2 J(\rho\delta e-\sigma_{lm}\delta\varepsilon_{lm})_{t=t_0}]\,\mathrm{d}D, \tag{93}$$

where * means a linear hereditary functional acting on LM and defined by (92), and δP_e is the part of the general performance functional having only the explicit design variations given by the following expression:

$$\delta P_e=\int_0^{t_0}\Bigg(\int_D\Bigg(\delta Jf+Jf_{;b}+\lambda_1^i[\delta(J\rho)\ddot{u}_i-\delta(J\rho F_i)]+\lambda_{1;l}^i\sigma_{kj}[\delta x_{i;s}d_{sj}Jd_{lk}+z_{i;s}\delta(d_{sj}Jd_{lk})]$$

$$+\lambda_2[\delta(J\rho)\dot{e}-\delta(J\rho h)+(q_i\delta(Jd_{ji}))_{;j}]-\delta J\sigma_{lm}\dot{\varepsilon}_{lm}+\lambda_3^{lm}[u_{l;q}\delta d_{qm}+u_{m;q}\delta d_{ql}+u_{k;p}u_{k;q}\delta(d_{pm}d_{ql})]$$

$$-\lambda_4^i D_b\overset{\infty}{\underset{\tau=0}{Q_i'}}\,\delta b-\lambda_5^{ij}D_b\overset{\infty}{\underset{\tau=0}{\Xi_{ij}}}\,\delta b-\lambda_6 D_b\overset{\infty}{\underset{\tau=0}{\Theta}}\,\delta b-\lambda_7 D_b\overset{\infty}{\underset{\tau=0}{\Psi}}\,\delta b\Bigg)\mathrm{d}D$$

$$+\int_{\Gamma_u}\left[\lambda_{1;l}^i d_{sj}\sigma_{kj}Jd_{lk}n_s-(\lambda_3^{im}d_{qm}+u_{i;p}\lambda_3^{lm}d_{qm}d_{pl})n_q-\lambda_4^n\left(\overset{\infty}{\underset{\tau=0}{Q_n'}}\right)_{;w_{ip}}n_p+Jf_{;w_{ip}}n_p\right]\delta\bar{u}_i$$

$$-\int_{\Gamma_s}\lambda_1^i\delta(J_\Gamma\bar{S}_i)\,\mathrm{d}\Gamma+\int_{\Gamma_q}\lambda_2[\delta(\bar{q}J_\Gamma)+q_i\delta(Jd_{ji})n_j]\,d\Gamma$$

$$+\int_{\Gamma_R}\lambda_2[\delta(RJ_\Gamma)(T_R^4-T^4)+4RJ_\Gamma T_R^3\delta T_R+q_i\delta(Jd_{ji})n_j]\,\mathrm{d}\Gamma$$

$$+\int_{\Gamma_h}\lambda_2[\delta(hJ_\Gamma)(T_f-T)+hJ_\Gamma\delta T_f+q_i\delta(Jd_{ji})n_j]\,\mathrm{d}\Gamma\Bigg)\mathrm{d}t+\int_D[J(f_{;\dot{u}_i}-\dot{f}_{;\ddot{u}_i}-\rho\dot{\lambda}_1^i)_{t=0}\delta u_{0i}$$

$$+J(f_{;\ddot{u}_i}+\rho\lambda_1^i)_{t=0}\delta\dot{u}_{0i}+\lambda_2 J(\rho\delta e-\sigma_{lm}\delta\varepsilon_{lm})_{t=0}]\,\mathrm{d}D. \tag{94}$$

To make the implicit variations disappear, the equations and boundary and terminal conditions for LM are obtained. The AVM equations are:

$$J\rho\ddot{\lambda}_1^i-(\lambda_{1;l}^i d_{sj}\sigma_{kj}Jd_{lk})_{;s}+(\lambda_3^{im}d_{qm}+\lambda_3^{lm}u_{i;p}d_{pl}d_{qm})_{;q}$$

$$+\left(\lambda_4^n D_{w_{ip}}\overset{\infty}{\underset{\tau=0}{Q_n'}}\,d_{sp}\right)_{;s}-(Jf_{;w_{ip}}d_{sp})_{;s}+J(f_{;u_i}-\dot{f}_{;\dot{u}_i}+\ddot{f}_{;\ddot{u}_i})=0, \tag{95}$$

$$J\frac{\mathrm{d}(\lambda_2\sigma_{ij})}{\mathrm{d}t}+\lambda_3^{ij}-\delta_{\varepsilon_{ij}}\overset{\infty}{\underset{\tau=0}{Q_n'^*}}(\lambda_4^n)-\delta_{\varepsilon_{ij}}\overset{\infty}{\underset{\tau=0}{\Xi_{st}^*}}(\lambda_5^{st})-\delta_{\varepsilon_{ij}}\overset{\infty}{\underset{\tau=0}{\Theta^*}}(\lambda_6)-\delta_{\varepsilon_{ij}}\overset{\infty}{\underset{\tau=0}{\Psi^*}}(\lambda_7)+Jf_{;\varepsilon_{ij}}=0, \tag{96}$$

$$\tfrac{1}{2}[\lambda_{1;l}^j d_{lk}+\lambda_{1;l}^k d_{lj}+\lambda_{1;l}^i u_{i;s}(d_{sj}d_{lk}+d_{sk}d_{lj})]J-J\lambda_2\dot{\varepsilon}_{kj}+\lambda_5^{kj}+Jf_{;\sigma_{kj}}=0, \tag{97}$$

$$-J\rho\dot{\lambda}_2+\lambda_7+Jf_{;e}=0, \tag{98}$$

$$-\delta_T\overset{\infty}{\underset{\tau=0}{Q_i'^*}}(\lambda_4^i)+\left(\delta_{g_m}\overset{\infty}{\underset{\tau=0}{Q_i'^*}}(\lambda_4^i)d_{pm}\right)_{;p}-\delta_T\overset{\infty}{\underset{\tau=0}{\Xi_{ij}^*}}(\lambda_5^{ij})-\delta_T\overset{\infty}{\underset{\tau=0}{\Theta^*}}(\lambda_6)-\delta_T\overset{\infty}{\underset{\tau=0}{\Psi^*}}(\lambda_7)$$

$$-\eta\lambda_7+Jf_{;T}=0, \tag{99}$$

$$-\lambda_{2;j}Jd_{ji}+\lambda_4^i+Jf_{;q_i}=0, \tag{100}$$

$$\lambda_6 - T\lambda_7 + Jf_{;\eta} = 0. \tag{101}$$

The boundary conditions are:

(1) On Γ_u

$$-\lambda_1^i + J_\Gamma f_{1;s_i} = 0; \tag{102}$$

(2) On Γ_s

$$\lambda^i_{1;l} d_{sj}\sigma_{kj} J d_{lk} n_s - (\lambda_3^{im} d_{qm} + u_{i;p}\lambda_3^{lm} d_{qm} d_{pl}) n_q - \lambda_4^n \left(\overset{\infty}{\underset{\tau=0}{Q'_n}}\right)_{;w_{ip}} d_{sp} n_s + J_\Gamma f_{2;u_i} + Jf_{;w_{ip}} d_{sp} n_s = 0; \tag{103}$$

(3) On Γ_h

$$-\delta_{g_m} \overset{\infty}{\underset{\tau=0}{Q'^*_i}}(\lambda_4^i) d_{pm} n_p - \lambda_2 J_\Gamma h + J_\Gamma f_{4;T} = 0; \tag{104}$$

(4) On Γ_R

$$-\delta_{g_m} \overset{\infty}{\underset{\tau=0}{Q'^*_i}}(\lambda_4^i) d_{pm} n_p - 4\lambda_2 J_\Gamma R T^3 + J_\Gamma f_{4;T} = 0; \tag{105}$$

(5) On Γ_q

$$-\delta_{g_m} \overset{\infty}{\underset{\tau=0}{Q'^*_i}}(\lambda_4^i) d_{pm} n_p + J_\Gamma f_{4;T} = 0; \tag{106}$$

(6) On Γ_T

$$\lambda_2 J d_{ji} n_j + J_\Gamma f_{3;q_i} = 0. \tag{107}$$

The terminal conditions are:

$$(f_{;\dot{u}_i} - \dot{f}_{;\ddot{u}_i} - \rho\dot{\lambda}_1^i)_{t=t_0} = 0, \tag{108}$$

$$(\rho\lambda_1^i + f_{;\ddot{u}_i})_{t=t_0} = 0, \tag{109}$$

$$(\lambda_2)_{t=t_0} = 0. \tag{110}$$

Then, variation of the general performance functional P will be:

$$\delta P = \delta P_e. \tag{111}$$

If the energy equation (11) is used, then attaching (70) and (71) via LM λ_2, λ_7 and going through similar derivations, adjoint equations (96), (97), (98), (99) and (101) are replaced by the following equations:

$$J\frac{d(\lambda_7\sigma_{ij})}{\mathrm{d}t} + \lambda_3^{ij} - \delta_{\varepsilon_{ij}} \overset{\infty}{\underset{\tau=0}{Q'^*_n}}(\lambda_4^n) - \delta_{\varepsilon_{ij}} \overset{\infty}{\underset{\tau=0}{\Xi^*_{st}}}(\lambda_5^{st}) - \delta_{\varepsilon_{ij}} \overset{\infty}{\underset{\tau=0}{\Theta^*}}(\lambda_6) - J\rho\delta_{\varepsilon_{ij}} \overset{\infty}{\underset{\tau=0}{\Psi^*}}(\dot{\lambda}_7) + Jf_{;\varepsilon_{ij}} = 0, \tag{112}$$

$$\tfrac{1}{2}[\lambda^i_{1;l} d_{lk} + \lambda^k_{1;l} d_{lj} + \lambda^i_{1;l} u_{i;s}(d_{sj} d_{lk} + d_{sk} d_{lj})]J + \lambda_5^{kj} + Jf_{;\sigma_{kj}} = 0, \tag{113}$$

$$\lambda_7 - \lambda_2 + f_{;\omega} = 0, \tag{114}$$

$$\lambda_2 J\rho\dot{\eta} - \delta_T \overset{\infty}{\underset{\tau=0}{Q_i'^*}}(\lambda_4^i) + \left(\delta_{g_m} \overset{\infty}{\underset{\tau=0}{Q_i'^*}}(\lambda_4^i) d_{pm}\right)_{;p} - \delta_T \overset{\infty}{\underset{\tau=0}{\Xi_{ij}^*}}(\lambda_5^{ij}) - \delta_T \overset{\infty}{\underset{\tau=0}{\Theta^*}}(\lambda_6)$$

$$- J\rho\delta_T \overset{\infty}{\underset{\tau=0}{\Psi^*}}(\dot{\lambda}_7) - J\rho \frac{d(\eta\lambda_7)}{dt} + Jf_{;T} = 0, \quad (115)$$

$$\lambda_6 - J\rho \frac{d(\lambda_2 T)}{dt} + J\rho\lambda_7 \dot{T} + Jf_{;\eta} = 0. \quad (116)$$

The boundary conditions stay the same, but one extra terminal condition should be satisfied in addition to (108)–(110):

$$(\lambda_7)_{t=t_0} = 0. \quad (117)$$

We see that the number of the AVM equations depends on the number of functionals for which sensitivities have to be calculated and is independent of the number of design variables which is consistent with what was said in the DDM section. Therefore, depending on the problem at hand, either AVM or DDM has to be chosen for the best benefits. We note also that AVM problem is a terminal value problem meaning that all primary field variables have to be known for every instant in time from 0 to t_0. This fact complicates numerical implementation since AVM sensitivities cannot be obtained along with the primary solution by marching in time, but rather have to be calculated after the complete primary solution was obtained producing massive computer memory requirements because the primary solution has to be stored at all time instants.

8. ANALYTICAL EXAMPLES

8.1. *Example 1: Thermoelastic medium, DDM*

We consider here design sensitivity calculations of a steady-state solution for an infinite thermoelastic medium subjected to a prescribed rate of heat generation similar to one described in Boley and Weiner (1960).

Infinitesimal strains and linear elastic Hooke's law are considered. The internal heat generation per unit volume Q is given by:

$$Q(x_1, t) = Q_1(t)\cos\frac{x_1}{L}, \quad (118)$$

where L is a wavelength parameter.

Appropriate constraints are applied so that only one component of the displacement field u_1 is non-zero. Also, typical of thermoelasticity with infinitesimal strains, it is assumed that $T/T_0 \ll 1$. Here and everywhere below T means difference between the actual absolute temperature and the initial absolute temperature.

The free energy function is (Nowacki, 1986):

$$\rho\psi = \rho\psi_0 + \left(\mu + \frac{\lambda}{2}\right)\varepsilon_{11}^2 - (3\lambda + 2\mu)\alpha\varepsilon_{11}T - \frac{\rho c}{2T_0}T^2, \quad (119)$$

where

λ, μ are the Lamé coefficients,
α is the coefficient of thermal expansion,
c is the specific heat at constraint strain.

The equations to be solved are:

$$\frac{\partial \sigma_{11}}{\partial x_1} = \rho \frac{\partial^2 u_1}{\partial t^2}, \tag{120}$$

$$\sigma_{11} = (\lambda+2\mu)\varepsilon_{11} - (3\lambda+2\mu)\alpha T, \tag{121}$$

$$\sigma_{22} = \sigma_{33} = \lambda\varepsilon_{11} - (3\lambda+2\mu)\alpha T, \tag{122}$$

$$\sigma_{12} = \sigma_{23} = \sigma_{13} = \varepsilon_{12} = \varepsilon_{23} = \varepsilon_{13} = \varepsilon_{22} = \varepsilon_{33} = 0, \tag{123}$$

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}, \tag{124}$$

$$\eta = \frac{cT}{T_0} + \alpha(3\lambda+2\mu)\frac{\varepsilon_{11}}{\rho}, \tag{125}$$

$$\rho T_0 \dot{\eta} = -\frac{\partial q_1}{\partial x_1} + \rho r, \tag{126}$$

$$q_1 = -k\frac{\partial T}{\partial x_1}. \tag{127}$$

Equations (120)–(127) are reduced to two equations to be solved for the displacement u_1 and the temperature T:

$$k\frac{\partial^2 T}{\partial x_1^2} - \rho c\frac{\partial T}{\partial t} - (3\lambda+2\mu)\alpha T_0 \frac{\partial^2 u_1}{\partial x_1 \partial t} + Q_1(t)\cos\frac{x_1}{L} = 0, \tag{128}$$

$$(\lambda+2\mu)\frac{\partial^2 u_1}{\partial x_1^2} - \rho\frac{\partial^2 u_1}{\partial t^2} - (3\lambda+2\mu)\alpha\frac{\partial T}{\partial x_1} = 0. \tag{129}$$

Introduce a nondimensional time variable $\tau = kt/\rho c L^2$ and assume $Q_1(\tau) = Q_0 \sin\tau$. The solutions then are taken in the following nondimensional form:

$$\frac{u_1}{L} = \frac{\rho c}{\alpha(3\lambda+2\mu)} H(\tau)\sin\frac{x_1}{L}, \tag{130}$$

$$\frac{T}{T_0} = G(\tau)\cos\frac{x_1}{L}. \tag{131}$$

For the unknown functions $H(\tau)$ and $G(\tau)$ obtain the following equations:

$$G + \frac{\mathrm{d}G}{\mathrm{d}\tau} + \frac{\mathrm{d}H}{\mathrm{d}\tau} = \frac{\sin\tau L^2}{T_0 k}, \tag{132}$$

$$H + K^2\frac{\mathrm{d}^2 F}{\mathrm{d}\tau^2} - \Delta G = 0, \tag{133}$$

where

$$K^2 = \frac{k^2}{\rho L^2 c^2(\lambda+2\mu)}, \tag{134}$$

$$\Delta = \frac{(3\lambda+2\mu)^2\alpha^2 T_0}{(\lambda+2\mu)\rho c}. \tag{135}$$

Solving (132), (133) for steady-state solutions, we obtain:

$$u_1(x_1,t) = \frac{L\rho c}{\alpha(3\lambda+2\mu)}(H_1 \sin\tau + H_2 \cos\tau)\sin\frac{x_1}{L}, \tag{136}$$

$$\frac{T(x_1,t)}{T_0} = (G_1 \sin\tau + G_2\cos\tau)\cos\frac{x_1}{L}, \tag{137}$$

where

$$H_1 = \frac{Q_0 L^2(1-K^2)\Delta}{[(1-K^2)^2+(1+\Delta-K^2)^2]kT_0}, \tag{138}$$

$$H_2 = -\frac{Q_0 L^2(1+\Delta-K^2)\Delta}{[(1-K^2)^2+(1+\Delta-K^2)^2]kT_0}, \tag{139}$$

$$G_1 = \frac{Q_0 L^2(1-K^2)^2}{[(1-K^2)^2+(1+\Delta-K^2)^2]kT_0}, \tag{140}$$

$$G_2 = -\frac{Q_0 L^2(1-K^2)(1+\Delta-K^2)}{[(1-K^2)^2+(1+\Delta-K^2)^2]kT_0}. \tag{141}$$

Since the problem is linear, the solution is a sum of the steady-state and transient solutions. For the same reason the design sensitivity solution is a sum of steady-state sensitivity solution and transient sensitivity solution. The steady-state part in the sensitivity solution corresponds to the steady-state part in the original solution, and, therefore, only the steady-state part in the original solution needs to be taken into account for sensitivity analysis since we are interested in the steady-state sensitivity.

The chosen design parameter is T_0. Then, the sensitivity equations in DDM are reduced to two equations for the displacement and temperature sensitivities $\tilde{u}_1$, and $\tilde{T}$:

$$k\frac{\partial^2\tilde{T}}{\partial x_1^2} - \rho c\frac{\partial\tilde{T}}{\partial t} - (3\lambda+2\mu)\alpha T_0\frac{\partial^2\tilde{u}_1}{\partial x_1\,\partial t} + \alpha(3\lambda+2\mu)\frac{\partial^2 u_1}{\partial x_1\,\partial t} = 0, \tag{142}$$

$$(\lambda+2\mu)\frac{\partial^2\tilde{u}_1}{\partial x_1^2} - \rho\frac{\partial^2\tilde{u}_1}{\partial t^2} - (3\lambda+2\mu)\alpha\frac{\partial\tilde{T}}{\partial x_1} = 0. \tag{143}$$

We note that except for the right-hand side (142) and (143) are not different from (128) and (129). Therefore, solving them similarly, we obtain the solutions for sensitivities:

$$\tilde{u}_1 = \frac{L\rho c}{\alpha(3\lambda+2\mu)}(A_1\sin\tau + A_2\cos\tau)\sin\frac{x_1}{L}, \tag{144}$$

$$\tilde{T} = T_0(B_1\sin\tau + B_2\cos\tau)\cos\frac{x_1}{L}, \tag{145}$$

where

$$A_1 = \frac{H_2(1-K^2)-H_1(1-K^2+\Delta)}{[(1-K^2)^2+(1-K^2+\Delta)^2]T_0}\Delta, \tag{146}$$

$$A_2 = -\frac{H_1(1-K^2)+H_2(1-K^2+\Delta)}{[(1-K^2)^2+(1-K^2+\Delta)^2]T_0}\Delta, \tag{147}$$

$$B_1 = \frac{H_2(1-K^2)^2-H_1(1-K^2+\Delta)(1-K^2)}{[(1-K^2)^2+(1-K^2+\Delta)^2]T_0}\Delta, \tag{148}$$

$$B_2 = \frac{-H_1(1-K^2)^2-H_2(1-K^2+\Delta)(1-K^2)}{[(1-K^2)^2+(1-K^2+\Delta)^2]T_0}\Delta. \tag{149}$$

Taking derivatives of H_1, H_2, G_1, G_2 with respect to T_0, we obtain

$$\frac{dH_1}{dT_0} = A_1,$$
$$\frac{dH_2}{dT_0} = A_2,$$
$$\frac{dG_1}{dT_0} = B_1,$$
$$\frac{dG_2}{dT_0} = B_2. \tag{150}$$

Hence,

$$\frac{du_1}{dT_0} = \tilde{u}_1,$$
$$\frac{dT}{dT_0} = \tilde{T}.$$

This verifies that the solutions of the sensitivity equations (142), (143) $\tilde{u}_1$, $\tilde{T}$ are the sensitivities of u_1, T.

8.2. *Example 2: Thermoviscoelastic medium, DDM*

Consider the same problem as in the previous example, but with viscoelastic material. We consider a linear thermoviscoelastic material with infinitesimal strains and with the free energy function proposed in Christensen and Naghdi (1967):

$$\begin{aligned}\rho\psi = \rho\psi_0 &+ \int_{-\infty}^{t} D_{ij}(t-\tau)\frac{\partial \varepsilon_{ij}}{\partial \tau}\,d\tau - \int_{-\infty}^{t} \lambda(t-\tau)\frac{\partial T(\tau)}{\partial \tau}\,d\tau \\ &+ \frac{1}{2}\int_{-\infty}^{t}\int_{-\infty}^{t} G_{ijkl}(t-\tau, t-s)\frac{\partial \varepsilon_{ij}(\tau)}{\partial \tau}\frac{\partial \varepsilon_{kl}(s)}{\partial s}\,d\tau\,ds \\ &- \int_{-\infty}^{t}\int_{-\infty}^{t} \Phi_{ij}(t-\tau, t-s)\frac{\partial \varepsilon_{ij}(\tau)}{\partial \tau}\frac{\partial T(s)}{\partial s}\,d\tau\,ds \\ &- \frac{1}{2}\int_{-\infty}^{t}\int_{-\infty}^{t} m(t-\tau, t-s)\frac{\partial T(\tau)}{\partial \tau}\frac{\partial T(s)}{\partial s}\,d\tau\,ds. \end{aligned} \tag{151}$$

The constitutive equations for stresses and entropy are:

$$\sigma_{ij} = D_{ij}(0)+\int_{-\infty}^{t} G_{ijkl}(t-\tau,0)\frac{\partial\varepsilon_{kl}(\tau)}{\partial\tau}\,d\tau-\int_{-\infty}^{t}\Phi_{ij}(0,t-\tau)\frac{\partial T(\tau)}{\partial\tau}\,d\tau, \tag{152}$$

$$\rho\eta = \lambda(0)+\int_{-\infty}^{t}\Phi_{ij}(t-\tau,0)\frac{\partial\varepsilon_{ij}(\tau)}{\partial\tau}\,d\tau+\int_{-\infty}^{t} m(t-\tau,0)\frac{\partial T(\tau)}{\partial\tau}\,d\tau. \tag{153}$$

Equations (120), (123), (124), (126) and (127) from the previous example do not change. Neglecting the second order terms, the current system of equations is reduced to two integrodifferential equations for the displacement u_1 and the temperature T:

$$\int_{-\infty}^{t} K(t-\tau)\frac{\partial^3 u_1}{\partial x_1^2\,\partial\tau}\,d\tau-\int_{-\infty}^{t}\phi(t-\tau)\frac{\partial^2 T}{\partial x_1\,\partial\tau}\,d\tau = \rho\frac{\partial^2 u_1}{\partial t^2}, \tag{154}$$

$$\phi(0)\frac{\partial^2 u_1}{\partial x_1\,\partial t}+\int_{-\infty}^{t}\alpha(t-\tau)\frac{\partial^2 u_1}{\partial x_1\,\partial\tau}\,d\tau+n(0)\frac{\partial T}{\partial\tau}+\int_{-\infty}^{t}\beta(t-\tau)\frac{\partial T}{\partial\tau}\,d\tau = \frac{k}{T_0}\frac{\partial^2 T}{\partial x^2}+\rho\frac{r}{T_0}, \tag{155}$$

where

$$K(t-\tau) = G_{1111}(t-\tau,0),\quad \phi(t-\tau) = \Phi_{11}(t-\tau,0),$$
$$\alpha = \frac{d\phi}{dt},\quad n(t-\tau) = m(t-\tau,0),\quad \beta = \frac{dn}{dt}.$$

Assuming that the internal heat source is given by $\rho r = \sin t\cos(x_1/L)$ we obtain the steady-state solutions:

$$u_1(x_1,t) = (H_1\sin t+H_2\cos t)\sin\frac{x}{L}, \tag{156}$$

$$T(x_1,t) = (G_1\sin t+G_2\cos t)\cos\frac{x}{L}, \tag{157}$$

where

$$H_1 = \frac{\xi_1 Q_0}{(\xi_1^2+\xi_2^2)T_0}, \tag{158}$$

$$H_2 = -\frac{\xi_2 Q_0}{(\xi_1^2+\xi_2^2)T_0}, \tag{159}$$

$$G_1 = \frac{(\rho+K_s)K_s+K_c^2}{L(K_s^2+K_c^2)}H_1+\frac{\rho}{L(K_s^2+K_c^2)}H_2, \tag{160}$$

$$G_2 = -\frac{\rho}{L(K_s^2+K_c^2)}H_1+\frac{(\rho+K_s)K_s+K_c^2}{L(K_s^2+K_c^2)}H_2, \tag{161}$$

$$\xi_1 = \alpha_s+\frac{\beta_s-\dfrac{k}{L^2T_0}}{K_s^2+K_c^2}[(\rho+K_s)K_s+K_c^2]+\rho\frac{n(0)+\beta_c}{K_s^2+K_c^2}, \tag{162}$$

$$\xi_2 = \alpha_c + \phi(0) - \rho \frac{\beta_s - \dfrac{k}{L^2 T_0}}{K_s^2 + K_c^2} + \frac{(n(0)+\beta_c)[(\rho + K_s)K_s + K_c^2]}{K_s^2 + K_c^2}, \tag{163}$$

with K_s, K_c, α_s, α_c, β_s and β_c being sine and cosine Fourier transforms of the respective relaxation functions with unit angular frequency.

The design parameter is the intensity of the internal heat source Q_0. Then, performing the Frechet differentiation and reducing the obtained sensitivity equations to two equations for the sensitivities of the displacement $\tilde{u}_1$ and the temperature $\tilde{T}$, we obtain:

$$\int_{-\infty}^{t} K(t-\tau)\frac{\partial^3 \tilde{u}_1}{\partial x_1^2 \partial \tau}\,d\tau - \int_{-\infty}^{t} \phi(t-\tau)\frac{\partial^2 \tilde{T}}{\partial x_1 \partial \tau}\,d\tau = \rho \frac{\partial^2 \tilde{u}_1}{\partial t^2}, \tag{164}$$

$$\phi(0)\frac{\partial^2 \tilde{u}_1}{\partial x_1 \partial t} + \int_{-\infty}^{t} \alpha(t-\tau)\frac{\partial^2 \tilde{u}_1}{\partial x_1 \partial \tau}\,d\tau + n(0)\frac{\partial \tilde{T}}{\partial \tau} + \int_{-\infty}^{t} \beta(t-\tau)\frac{\partial \tilde{T}}{\partial \tau}\,d\tau = \frac{k}{T_0}\frac{\partial^2 \tilde{T}}{\partial x_1^2} + \frac{(\rho r)'}{T_0}, \tag{165}$$

where $(\rho r)'$ is $\partial(\rho r)/\partial Q_0$.

Again, the sensitivity equations are identical to the primary equations with the exception of the right-hand side due to the linearity of the primary equations.

Solving the above equations, we obtain:

$$\tilde{u}_1(x_1, t) = (\tilde{H}_1 \sin t + \tilde{H}_2 \cos t)\sin\frac{x_1}{L}, \tag{166}$$

$$\tilde{T}(x_1, t) = (\tilde{G}_1 \sin t + \tilde{G}_2 \cos t)\cos\frac{x_1}{L}, \tag{167}$$

with $\tilde{H}_1$, $\tilde{H}_2$, $\tilde{G}_1$ and $\tilde{G}_2$ being given by the following expression:

$$\tilde{H}_1 = \frac{\xi_1}{(\xi_1^2 + \xi_2^2)T_0}, \tag{168}$$

$$\tilde{H}_2 = -\frac{\xi_2}{(\xi_1^2 + \xi_2^2)T_0}, \tag{169}$$

$$\tilde{G}_1 = \frac{(\rho + K_s)K_s + K_c^2}{L(K_s^2 + K_c^2)}\tilde{H}_1 + \frac{\rho}{L(K_s^2 + K_c^2)}\tilde{H}_2, \tag{170}$$

$$\tilde{G}_2 = -\frac{\rho}{L(K_s^2 + K_c^2)}\tilde{H}_1 + \frac{(\rho + K_s)K_s + K_c^2}{L(K_s^2 + K_c^2)}\tilde{H}_2. \tag{171}$$

It is clear from (158)–(161) and (168)–(171) that

$$\tilde{H}_1 = \frac{dH_1}{dQ_0},$$

$$\tilde{H}_2 = \frac{dH_2}{dQ_0},$$

$$\tilde{G}_1 = \frac{dG_1}{dQ_0},$$

$$\tilde{G}_2 = \frac{dG_2}{dQ_0}, \tag{172}$$

and, therefore, the obtained $\tilde{u}_1$ and $\tilde{T}$ are sensitivities of u_1 and T.

8.3. *Example 3: Thermoelastic medium, AVM*

We consider the same problem as in example 1, but a quasistatic one meaning that the magnitude of the inertial term is small compared to the two other terms in eqn (129), and it can be dropped. Equation (128) does not change. Now we are considering the interval $[0, L]$ along the x_1 axis. The boundary conditions are:

$$u_1(0) = u_1(L) = 0; \quad q_1(0) = q_1(L) = 0. \tag{173}$$

Consider the following expression for the heat source:

$$\rho r = Q_0 \sin t \cos \frac{\pi x_1}{L}. \tag{174}$$

The initial condition is:

$$T(0) = 0. \tag{175}$$

We express the displacement field and the temperature field as:

$$u_1 = H(t) \sin \frac{\pi x_1}{L},$$
$$T = G(t) \cos \frac{\pi x_1}{L}. \tag{176}$$

For the functions H and G, we obtain equations:

$$\xi \dot{H} + \zeta H = Q_0 \sin t,$$
$$G = \chi H, \tag{177}$$

where

$$\xi = \rho c \frac{\pi(\lambda+2\mu)}{L\alpha(3\lambda+2\mu)} + (3\lambda+2\mu)\frac{\pi}{L}\alpha T_0,$$
$$\zeta = k \frac{\pi^3(\lambda+2\mu)}{L^3\alpha(3\lambda+2\mu)},$$
$$\chi = \frac{\pi(\lambda+2\mu)}{L\alpha(3\lambda+2\mu)}.$$

Solving eqns (177), we get:

$$H = H_1(\cos t - e^{-(\zeta/\xi)t}) + H_2 \sin t,$$
$$G = \chi H_1(\cos t - e^{-(\zeta/\xi)t}) + \chi H_2 \sin t, \tag{178}$$

where

$$H_1 = -\frac{\xi Q_0}{\xi^2+\zeta^2},$$

$$H_2 = \frac{\zeta Q_0}{\xi^2+\zeta^2}. \tag{179}$$

To illustrate an application of AVM we will calculate sensitivity of the displacement u_1 at the coordinate $x_1 = y$ and point in time $t = \tau$ with respect to the design parameter Q_0. In such a case the general performance functional becomes:

$$P = \int_0^{t_0} \delta(t-\tau)\left[\int_0^L \delta(x_1-y)u_1(x_1,t)\,\mathrm{d}x_1\right]\mathrm{d}t. \tag{180}$$

The adjoint equations in the case of the infinitesimal strains cannot be obtained by just dropping the higher order terms like in the case of a nonlinear continuum. The entire process of derivation of the adjoint equations has to be repeated in order to arrive at the sought equations. The adjoint equations obtained in our case are:

$$\frac{\partial\lambda_3}{\partial x_1}+\delta(t-\tau)\delta(x_1-y) = 0,\quad \lambda_3-\lambda_5(\lambda+2\mu)-\lambda_6\frac{\alpha(3\lambda+2\mu)}{\rho} = 0,$$

$$\frac{\partial\lambda_1}{\partial x_1}+\lambda_5 = 0,\quad -k\frac{\partial\lambda_4}{\partial x_1}+\lambda_5(3\lambda+2\mu)\alpha-\lambda_6\frac{c}{T_0} = 0,$$

$$-\frac{\partial\lambda_2}{\partial x_1}+\lambda_4 = 0,\quad \lambda_6-\rho T_0\dot{\lambda}_2 = 0. \tag{181}$$

Equations (181) can be reduced to two equations for λ_1 and λ_2:

$$(\lambda+2\mu)\frac{\partial^2\lambda_1}{\partial x_1^2}-\alpha(3\lambda+2\mu)T_0\frac{\partial^2\lambda_2}{\partial t\,\partial x_1}-\delta(t-\tau)\delta(x_1-y) = 0,$$

$$k\frac{\partial^2\lambda_2}{\partial x_1^2}+\alpha(3\lambda+2\mu)\frac{\partial\lambda_1}{\partial x_1}+\rho c\frac{\partial\lambda_2}{\partial t} = 0. \tag{182}$$

It is clear, however, from (93) and (176) that (182) can be written in a weak form:

$$(\lambda+2\mu)\int_0^L\frac{\partial^2\lambda_1}{\partial x_1^2}\sin\frac{\pi x_1}{L}\,\mathrm{d}x_1-\alpha(3\lambda+2\mu)T_0\int_0^L\frac{\partial^2\lambda_2}{\partial t\,\partial x_1}\sin\frac{\pi x_1}{L}\,\mathrm{d}x_1-\delta(t-\tau)\sin\frac{\pi y}{L} = 0,$$

$$k\int_0^L\frac{\partial^2\lambda_2}{\partial x_1^2}\cos\frac{\pi x_1}{L}\,\mathrm{d}x_1+\alpha(3\lambda+2\mu)\int_0^L\frac{\partial\lambda_1}{\partial x_1}\cos\frac{\pi x_1}{L}\,\mathrm{d}x_1+\rho c\int_0^L\frac{\partial\lambda_2}{\partial t}\cos\frac{\pi x_1}{L}\,\mathrm{d}x_1 = 0. \tag{183}$$

Expressing λ_1 and λ_2 as

$$\lambda_1 = \beta_1\sin\frac{\pi x_1}{L},$$

$$\lambda_2 = \beta_2\cos\frac{\pi x_1}{L}, \tag{184}$$

and substituting (184) into (183), we obtain equations for β_1 and β_2:

$$\beta_1 = \xi_1\beta_2 - \xi_2\dot{\beta}_2, \quad \xi_3\beta_1 - \xi_4\dot{\beta}_2 + \delta(t-\tau)\sin\frac{\pi y}{L} = 0, \tag{185}$$

where

$$\xi_1 = \frac{k\pi}{L\alpha(3\lambda+2\mu)},$$

$$\xi_2 = \frac{L\rho c}{\pi\alpha(3\lambda+2\mu)},$$

$$\xi_3 = \frac{\pi^2(\lambda+2\mu)}{2L},$$

$$\xi_4 = \frac{\pi\alpha(3\lambda+2\mu)T_0}{2}.$$

Equations (185) have to be solved with a terminal condition:

$$\beta_2(t_0) = 0. \tag{186}$$

Solving equation (185) with the terminal condition (186), we obtain the solution for β_2:

$$\beta_2 = \begin{cases} -\dfrac{\sin\dfrac{\pi y}{L}}{\kappa} e^{\gamma(t-\tau)}, & \text{if } t < \tau, \\ 0, & \text{if } t \geqslant \tau, \end{cases} \tag{187}$$

where

$$\kappa = \xi_2\xi_3 + \xi_4; \quad \gamma = \frac{\xi_1\xi_3}{\kappa}.$$

Sensitivity $\tilde{u}_1$ of the displacement u_1 with respect to the design parameter Q_0 can be calculated now by utilizing (111) and (94). In the present case, the formula becomes:

$$\tilde{u}_1 = -\int_0^{t_0}\left[\int_0^L \lambda_2 \frac{\partial(\rho r)}{\partial Q_0}\,\mathrm{d}x_1\right]\mathrm{d}t. \tag{188}$$

Substituting (174), (184) and (187) into (188), we obtain:

$$\tilde{u}_1 = \frac{L}{2(\gamma^2+1)\kappa}[\gamma\sin\tau - \cos\tau + \mathrm{e}^{-\gamma\tau}]\sin\frac{\pi y}{L}. \tag{189}$$

Calculating the derivative with respect to Q_0 of the displacement u_1 given by (176), we observe that

$$\tilde{u}_1 = \frac{\mathrm{d}u_1}{\mathrm{d}Q_0}, \tag{190}$$

which verifies AVM.

9. CONCLUSION

The formulation for design sensitivity analysis of transient dynamic, arbitrarily nonlinear fully coupled thermoviscoelastic systems was presented. The constitutive representation was based on the description of simple thermoviscoelastic materials in irreversible thermodynamics. A performance functional allowing to calculate sensitivities of the quantities on either deformed or undeformed configurations was used. Domain parametrization or reference volume concept was used for unification of shape and nonshape design sensitivities. Both DDM and AVM sensitivity equations were obtained. As usual, depending on the problem at hand, either of the methods may be advantageous. However, for transient problems the DDM is probably superior since an initial value problem is solved contrary to a terminal value problem in the AVM. Simple analytical examples for DDM as well as for AVM were solved illustrating and verifying the developed equations. The presented development provides a good starting point for the discretization of the developed equations and establishing the relations between the primary structural analysis and the design sensitivity analysis for subsequent numerical implementation.

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